A novel method for the forced vibrations of nonlinear oscillators

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ABSTRACT — *A novel method named parameter-split-multiple-scales method is proposed to analyze the forced vibrations of strongly nonlinear oscillators. A nonlinear oscillator equivalent to the original oscillator is formulated by introducing some unknown parameters first. After that, the approximated analytical solution to the equivalent nonlinear oscillator is obtained by classical multiple-scales method. In the last, the split parameters are determined by minimizing the residual error of the original nonlinear oscillator equation. The solution procedure about the forced vibration of a Duffing oscillator by the proposed method is presented in this paper. Two strongly nonlinear Duffing oscillators are considered as illustrative examples to test the feasibility of the parameter-split-multiple-scales method. By the comparison of the frequency response curves obtained by classical multiple-scales method, the parameter-split-multiple-scales method and numerical continuation method, the advantages and the effectiveness of the proposed method are presented.*

1 Introduction

Nonlinear system can exhibit many phenomena that are different from or can't be found in linear systems [1]. However, it is difficult or impossible to find the exact solutions to the nonlinear systems since they are expressed by nonlinear ordinary differential equations (ODEs) or nonlinear partial differential equations (PDEs). With the increasing interests in the applications of nonlinear problems, various analytical methods for finding the approximate analytical solutions to those nonlinear ODEs have been developed in recent years. The perturbation method is one of the approximate analytical methods. The perturbation method breaks a nonlinear equation into some linear equations which exact solutions are obtainable and can be solved one by one. The multiple-scales (MS) method is a representative of the perturbation methods which is well known for its eliminating secular terms. The MS method has been applied to many oscillation problems such as the vibrations of cables, vibrations of beams and plates, etc [2, 3, 4]. However, due to the requirement for small parameter in the system, the MS method lost its validity when the problem is strongly nonlinear [5]. To solve strongly nonlinear oscillation problems, a lot of approximate analytical methods have been developed in recent years [6, 7, 8, 9, 10, 11, 12].

In this paper, a novel method is proposed for optimizing the nonlinear oscillator solution obtained by the MS method. The strategy of this method is that some parameters in the oscillator are split by introducing some unknown parameters. Based on the solution obtained by the MS method, an optimization objective is formulated and the introduced unknown parameters are determined by minimizing the cumulative residual error of the original oscillation equation. Hence the method is named parameter-split-multiple-scales (PSMS) method. The Duffing oscillator with viscous damping and harmonic external force [13] is adopted to test the effectiveness of the proposed method. The frequency response curves (FRCs) obtained by the conventional MS method, the PSMS method and numerical continuation method (NCM) [14] are compared with each other. The results show that the solutions obtained by the PSMS method are much improved comparing to those obtained by the MS method. The FRCs obtained by the PSMS method are verified by the FRCs obtained by the NCM.

2 Parameter-split-multiple-scales method

Consider the following non-dimensional Duffing oscillator

$$\ddot{y}+c\varepsilon^2\dot{y}+\omega_0^2 y+\eta\varepsilon y^3=F\varepsilon^2\cos(\Omega t). \tag{1}$$

This oscillator is a damped and harmonically forced Duffing oscillator that can be found in many applications such as the forced vibrations of pendulum, isolator, the vibrations of nonlinear beam and plate, electrical circuit, etc[13]. The Duffing oscillator was also popularly analyzed for examining new solution procedures. Therefore, the Duffing oscillator is selected as an example to test the effectiveness of the proposed method.

2.1 Parameter splitting (PS)

The natural frequency ω_0 and the nonlinear parameter η are split and expressed as

$$\omega_0^2=\omega_{00}^2+\omega_{01}^2\varepsilon+\omega_{02}^2\varepsilon^2 \tag{2}$$

and

$$\eta=\eta_1+\eta_2\varepsilon. \tag{3}$$

Then the Duffing oscillator can be rewritten as

$$\ddot{y}+c\varepsilon^2\dot{y}+\omega_{00}^2 y+\omega_{01}^2\varepsilon y+\omega_{02}^2\varepsilon^2 y+\eta_1\varepsilon y^3+\eta_2\varepsilon^2 y^3=F\varepsilon^2\cos(\Omega t) \tag{4}$$

where ω_{01}, ω_{02} and η_2 are split parameters to be determined by minimizing the residual error to be introduced in the following.

2.2 Solution to the equivalent oscillator by multiple-scales method

Since the secular terms can be eliminated by the MS method and the MS method is not effective for analyzing the strongly nonlinear oscillators [12], the MS method is adopted to analyze the equivalent Duffing oscillator presented in Eq. (4) to examine the effectiveness of the proposed procedures. With the MS method, the response of the equivalent oscillator is assumed to be

$$y=y_0(T_0,T_1,T_2)+\varepsilon y_1(T_0,T_1,T_2)+\varepsilon^2 y_2(T_0,T_1,T_2)+O(\varepsilon^3) \tag{5}$$

where T_0, T_1 and T_2 are the fast and slow time scales expressed by

$$T_0=t,\ T_1=\varepsilon t,\ T_2=\varepsilon^2 t. \tag{6}$$

By chain rule, the operators of time derivatives are

$$\frac{d}{dt} = D_0+\varepsilon D_1+\varepsilon^2 D_2+\ldots, \tag{7}$$

$$\frac{d^2}{dt^2} = D_0^2+2\varepsilon D_0 D_1+\varepsilon^2(D_1^2+2D_0D_2)+\ldots, \tag{8}$$

where $D_n=\partial/\partial T_n$ and $D_n^2=\partial^2/\partial T_n^2$. Substituting Eqs. (5), (7) and (8) into Eq. (4) and equating the coefficients of $\varepsilon^m(m=0,1,2)$ to zero lead to the following equations.

$$O(\varepsilon^0): \qquad D_0^2(y_0)+\omega_{00}^2 y_0=0, \tag{9}$$

$$O(\varepsilon^1): \qquad D_0^2(y_1)+\omega_{00}^2 y_1=-2D_0D_1y_0-\omega_{01}^2 y_0-\eta_1 y_0^3, \tag{10}$$

$$O(\varepsilon^2): \qquad D_0^2(y_2)+\omega_{00}^2 y_2=F\cos(\Omega t)-2D_0D_1y_1-D_1^2y_0-2D_0D_2y_0-cD_0y_0-\omega_{01}^2 y_1-\omega_{02}^2 y_0 \\ -3\eta_1 y_0^2 y_1-\eta_2 y_0^3. \tag{11}$$

The $O(\varepsilon^0)$ equation is a homogenous differential equation, the solution to it is

$$y_0 = C(T_1,T_2)e^{i\omega_{00}T_0} + \bar{C}(T_1,T_2)e^{-i\omega_{00}T_0} \tag{12}$$

where C is a function of time scales T_1 and T_2 which can be determined by omitting the secular terms in the $O(\varepsilon^1)$ equation. Substituting Eq. (12) into the righthand side of the $O(\varepsilon^1)$ equation and eliminating the secular terms yield

$$3\eta_1 C^2\bar{C}e^{i\omega_{00}T_0} + 2iD_1(C)\omega_{00}e^{i\omega_{00}T_0} + C\omega_{01}^2 e^{i\omega_{00}T_0} = 0 \tag{13}$$

and

$$y_1 = B_{13}e^{3i\omega_{00}T_0} + \bar{B}_{13}e^{-3i\omega_{00}T_0}, \tag{14}$$

in which

$$B_{13} = \frac{\eta_1 C^3}{8\omega_{00}^2}. \tag{15}$$

Substituting the expressions of y_0 and y_1 into the $O(\varepsilon^2)$ equation, eliminating the secular terms, and using the expression $\Omega = \omega_{00} + \varepsilon^2\sigma$ where σ is a detuning parameter that can be determined if Ω is given, it gives

$$D_2(C) = \frac{Fe^{i\sigma T_2}}{4i\omega_{00}} - \frac{cC}{2} - \frac{C\omega_{02}^2}{2i\omega_{00}} + \frac{C\omega_{01}^4}{8i\omega_{00}^3} - \frac{3\eta_2 C^2\bar{C}}{2i\omega_{00}} + \frac{3\eta_1 C^2\bar{C}\omega_{01}^2}{4i\omega_{00}^3} + \frac{15\eta_1^2 C^3\bar{C}^2}{16i\omega_{00}^3} \tag{16}$$

and

$$y_2 = B_{23}e^{3i\omega_{00}T_0} + B_{25}e^{5i\omega_{00}T_0} + \bar{B}_{23}e^{-3i\omega_{00}T_0} + \bar{B}_{25}e^{-5i\omega_{00}T_0}, \tag{17}$$

in which

$$B_{23} = \frac{\eta_2 C^3}{8\omega_{00}^2} - \frac{\eta_1 C^3\omega_{01}^2}{8\omega_{00}^4} - \frac{21\eta_1^2 C^4\bar{C}}{64\omega_{00}^4}, \tag{18}$$

and

$$B_{25} = \frac{\eta_1^2 C^5}{64\omega_{00}^4}. \tag{19}$$

The time derivative of C can be expressed as

$$\frac{dC}{dt} = \varepsilon D_1(C) + \varepsilon^2 D_2(C) + O(\varepsilon^3). \tag{20}$$

The polar form of C is assumed to be

$$C = \frac{1}{2}Ae^{ib}, \tag{21}$$

where A is the response amplitude and b is the phase of oscillator response. Substituting Eqs. (13), (16) and (21) into Eq. (20) and separating the real and imaginary parts yield

$$\dot{A} = \frac{F\varepsilon^2}{2\omega_{00}}\sin\gamma - \frac{cA\varepsilon^2}{2} \tag{22}$$

and

$$\dot{\gamma} = \Omega - \omega_{00} - \frac{\varepsilon\omega_{01}^2}{2\omega_{00}} - \frac{\varepsilon^2\omega_{02}^2}{2\omega_{00}} + \frac{\varepsilon^2\omega_{01}^4}{8\omega_{00}^3} - \frac{3A^2\eta_2\varepsilon^2}{8\omega_{00}} + \frac{15A^4\eta_1^2\varepsilon^2}{256\omega_{00}^3} - \frac{3A^2\eta_1\varepsilon}{8\omega_{00}} + \frac{\varepsilon^2 F\cos(\gamma)}{2A\omega_{00}} + \frac{3A^2\eta_1\varepsilon^2\omega_{01}^2}{16\omega_{00}^3}, \tag{23}$$

where $\gamma = \sigma T_2 - b$.

For steady state, $\dot{A}$ and $\dot{\gamma}$ are equal to zero. Then the frequency-response curve can be obtained by eliminating γ

and σ in Eq. (23). The relation between the excitation frequency and the response amplitude at steady state is then obtained to be

$$\Omega = \omega_{00} + \frac{\varepsilon\omega_{01}^2}{2\omega_{00}} + \frac{\varepsilon^2\omega_{02}^2}{2\omega_{00}} - \frac{\varepsilon^2\omega_{01}^4}{8\omega_{00}^3} + \frac{3A^2\eta_2\varepsilon^2}{8\omega_{00}} - \frac{15A^4\eta_1^2\varepsilon^2}{256\omega_{00}^3} + \frac{3A^2\eta_1\varepsilon}{8\omega_{00}} - \frac{\varepsilon^2 F\cos(\gamma)}{2A\omega_{00}} - \frac{3A^2\eta_1\varepsilon^2\omega_{01}^2}{16\omega_{00}^3}. \quad (24)$$

The approximate response of the oscillator is obtained to be

$$y_a = \mathbb{A}_1\cos(\Omega t-\gamma) + 2\mathbb{A}_3\cos[3(\Omega t-\gamma)] + 2\mathbb{A}_5\cos[5(\Omega t-\gamma)] \quad (25)$$

in which

$$\mathbb{A}_1 = A, \quad (26)$$

$$\mathbb{A}_3 = \frac{\eta A^3}{8\omega_{00}^2} - \frac{\eta_1\omega_{01}^2\varepsilon^2 A^3}{8\omega_{00}^4} - \frac{21\eta_1^2\varepsilon^2 A^5}{64\omega_{00}^4}, \quad (27)$$

and

$$\mathbb{A}_5 = \frac{\eta_1^2\varepsilon^2 A^5}{64\omega_{00}^4}. \quad (28)$$

2.3 Optimization objective

From Eqs. (25)- (28) it is seen that the expression of y_a can be considered as a function $f(\omega_{01},\omega_{02},\eta_1)$ of the variables ω_{01}, ω_{02} and η_1 once the system parameters ω_0, c, η, Ω and F are given. Select an interval $\boldsymbol{\Omega} = [\Omega_l,\Omega_u]$ on the positive frequency axis, in which most of the frequency-response curve falls. Then for a given value of excitation frequency within $\boldsymbol{\Omega}$, the values of ω_{01}, ω_{02} and η_1 can be determined by minimizing the value of the residual error R_e given as

$$R_e = \int_0^T \left[\ddot{y}_a + c\varepsilon^2\dot{y}_a + \omega_0^2 y_a + \eta\varepsilon y_a^3 - F\varepsilon^2\cos(\Omega t)\right]^2 dt \quad (29)$$

where $T = 2\pi/\Omega$. Since the function R_e consists of periodic functions with periods $\frac{2\pi}{n\Omega}$ $(n = 1,2,3\ldots)$ where Ω is the excitation frequency, the integration upper limit is hence selected as T to cumulate all the errors induced by each periodic function. The nonlinear oscillation problem, therefore, has been converted to an optimization problem. The unknown splitting parameters $\omega_0 1$, $\omega_0 2$ and η_2 are determined by minimizing R_e via the Levenberg-Marquardt algorithm (LMA). The complete FRC can then be obtained by repeating this procedure and varying Ω from Ω_l to Ω_u.

3 Illustrative exmaples

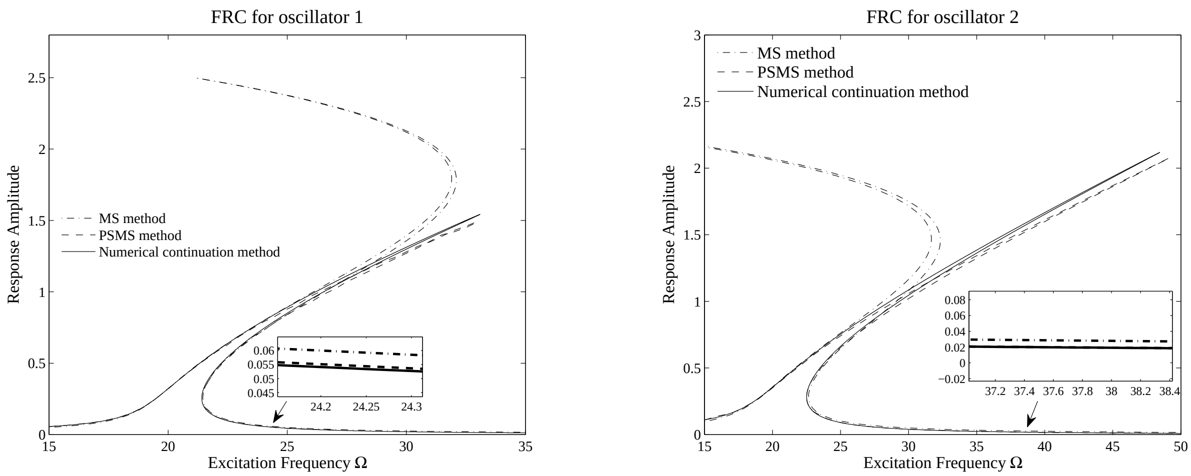
Two strongly nonlinear Duffing oscillators whose ratios of nonlinear stiffness/linear stiffness ($\frac{\eta\varepsilon y^3}{\omega_0^2 y}$) are equal to y^2 and $2y^2$, respectively, are analyzed to examine the feasibility of the proposed method in the next Section.

The parameter values of the oscillators are listed in Tab. 1. The FRCs obtained by the PSMS method,

Tab. 1: Parameter values in the Duffing oscillators

Oscillator	ε	ω_0	c	η	F
1	0.1	20	20	4,000	1,000
2	0.1	20	20	6,000	2,000

conventional MS method and numerical continuation method are shown and compared in Figs. 1(a) and 1(b). A larger redundant portion can be observed in each FRC obtained by MS method in comparison with those obtained by NCM as the ratio of $\eta\varepsilon y^3/\omega_0^2 y$ becomes larger. However, even when the response amplitude is small, the



(a) $\varepsilon = 0.1, \omega_0 = 20, c = 20, \eta = 4,000$ and $f = 1,000$

(b) $\varepsilon = 0.1, \omega_0 = 20, c = 20, \eta = 6,000$ and $f = 2,000$

Fig. 1: The FRCs obtained by the MS method, the PSMS method and the numerical continuation method.

solutions obtained by the MS method still deviate more from the solutions obtained by the numerical continuation method than those obtained by the PSMS method do, which can be seen from the 'zoomed-in' figures shown in Figs. 1(a) and 1(b). This phenomenon is caused by the truncation error in Eq. 5. On the contrary, the PSMS method can give more accurate solutions to the Duffing oscillators in the whole frequency domain.

4 Conclusions

The strongly nonlinear Duffing oscillator with viscous damping and harmonic force is analyzed by a novel method named PSMS method in this paper. The FRCs obtained by the PSMS method are compared with those obtained by conventional MS method and examined by the FRCs obtained by numerical method. The results show that the FRCs obtained by the PSMS method are much improved comparing to those obtained by the MS method. The error in the FRCs obtained by the MS method increases as the response amplitude increases. The solutions obtained by the MS method are not acceptable even when the response amplitude is small. The procedure presented in this paper is not limit to the Duffing oscillator. Other oscillators can also be analyzed by this method. It is seen that this procedure is not limited to improving the solutions obtained by the multiple-scales method. The solutions from other perturbation methods can also be improved by this procedure.

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