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A rigid body formulation with non-redundant unified local velocity coordinates

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ABSTRACT — *In this paper, we present a formulation for spatial rigid body motion based on a set of non-redundant, homogeneous local velocity coordinates. In contrast to the common practice, the proposed approach renounces the distinction between translational and angular velocity in the sense that it only makes use of translational local velocities. To obtain the new set of coordinates, we use the velocity vectors of six properly selected points of a rigid body, and represent the rigid body's translational and rotational velocities in terms of six scalar local translational velocities. Importantly, the equations of motion are derived without making use of the rotation matrix or the angular velocity vector. The position coordinates and the orientation of the body are computed by means of the exponential map on the special Euclidean group $SE(3)$. In order to solve the incremental motion vector differential equation, we introduce the inverse of the tangent operator corresponding to our local velocity coordinates. Furthermore, we use a recently presented 4th-order Runge-Kutta Lie group time integration scheme, which we extend such that it can be used with respect to our approach and elements of $SE(3)$. To show the applicability of our approach we simulate the unstable rotation of a rigid body.*

1 Introduction

In this paper, we present a formulation for spatial rigid body motion based on a set of non-redundant, homogeneous local (translational) velocity coordinates. The initial motivation for the present work was the investigation of local velocity coordinates of a first order H(curl)-conforming finite element proposed by Nédélec [1], which turned out to be relevant for rigid bodies as well, because the latter finite element offers six non-redundant and – regarding their physical interpretation – equivalent (=unified) displacement coordinates as degrees of freedom. Since the spatial motion of a rigid body is known, if the position and velocity (three translational velocity coordinates) of one of its points as well as its angular velocity (three angular velocity coordinates) are known [2], the questions arises, if it is possible to describe a rigid body motion using six unified local velocity coordinates, which correspond to the displacement coordinates of this first order H(curl)-conforming finite element and whether and how the resulting equations can be solved.

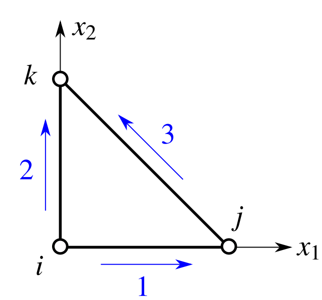
Modelling mechanical systems in such a way that they can be efficiently simulated plays an important role in many areas, e.g. in real-time simulation and control [3]. The number, complexity and computational efficiency of the equations describing the system strongly depend on the choice of coordinates used to model the mechanical system [3, 4]. A common approach to describe the kinematics of a multibody system is to divide the general rigid body motion into a translational motion and a rotational motion [2, 5, 6, 7], as is the case with the Newton-Euler equations, for example.

Six coordinates are at least required to define the configuration of a rigid body in space [5]. Three Cartesian coordinates usually define the origin of the body's reference frame [5]. In engineering applications, e.g. rotor dynamics, the body's orientation is often characterized by three angles of rotation like Euler angles or Bryant angles [8]. However, in case of large rotations, the representation of the body's orientation in terms of three angles of rotation suffers from singularities [6]. Alternative representations, which provide a singularity free description of the body's orientation either use for example the components of a rotation matrix or Euler parameters as degrees of freedom [2].

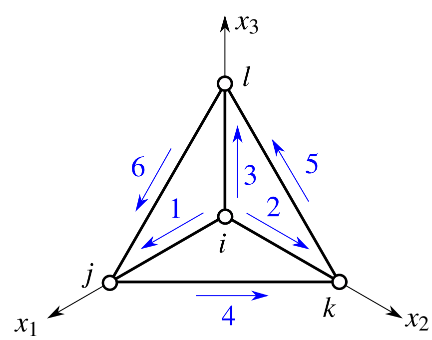
An alternative set of coordinates has been presented by Garcia de Jalon and co-workers in the early eighties [9]. They created and further developed the natural coordinates to describe the 2-D and 3-D motion of multibody systems [10]. The natural coordinates approach uses Cartesian coordinates of three (or more) non-aligned points to define the position of a solid in space [11]. Natural coordinates require neither angles nor any form of angle parametrization to describe the orientation of a body in space, which leads to constant inertia matrices and a simple form of the equations of motion [11]. Natural coordinates can be seen as a set of dependent parameters used to describe rotations [11]. In contrast to the coordinates used in [5], the description of the motion of a multibody system with the natural coordinates yields a unified form of the descriptive equations of motion but at the expense of the use of non redundant position coordinates.

A description of the motion of a multibody system based on inhomogeneous but non-redundant coordinates at velocity level is possible by describing the kinematics using Lie groups and the associated Lie algebras. The kinematics are currently described either with the Lie group formed by the Cartesian product of the group of translations (Euclidean space) with the special orthogonal group SO(3) or with the special Euclidean group SE(3) [12, 13]. The integration of the body-fixed velocity coordinates, which we will also refer to in the following as local velocity coordinates, is possible with the help of the matrix exponential function. Crouch and Grossmann [14] and Munthe-Kaas [15, 16] developed generalizations of classical Runge-Kutta methods and multi-step time integration algorithms to solve differential equations on Lie groups. Brüls and Cardona [17] exploited the classical generalized-α method proposed by Chung and Hulbert [18] for time integration on Lie groups which was further developed by Arnold et al. [8]. However, still the rigid body motion is partitioned into a translational and a rotational motion.

Nédélec introduced two new families of mixed finite elements in $\mathbb{R}^3$, which are conforming on the Sobolev spaces H(div) and H(curl) [1]. Those element families are commonly used to approximate solutions of Maxwell's equations, the equations of elasticity or stokes equations [19]. In particular, the lowest order H(curl) conforming element family, which consists of Nédélec elements of 1st-kind, is of interest for this work, see Fig. 1. In the three dimensional case, this finite element offers six non-redundant – and regarding their physical interpretation – equivalent degrees of freedom, which are associated with the edges of the mesh [19]. This is illustrated by Fig. 1b, showing the six local displacement coordinates. That means, the motion of a rigid body is fully described



(a) 2D Nédélec element (lowest order k=1: #edges=3, #dofs=3)



(b) 3D Nédélec element (lowest order k=1: #edges=6, #dofs=6)

Fig. 1: Degrees of freedom of a Nédélec element of 1st-kind.

by using only translational velocity coordinates. Moreover, we want to point out, that these translational velocity coordinates are fully equivalent (=unified) regarding their physical interpretation, which means that no partition into translational and rotational parts is introduced. Since the degrees of freedom in the Nédélec element of 1st-kind represent coordinates which are defined in a body-fixed frame, Lie group time integrators based on the exponential map such as e.g. the ones presented by Crouch and Grossmann [14] or Munthe-Kaas [15, 16] could be applied to obtain information about the position and orientation of rigid body.

The rest of the paper is organized as follows. In Sect. 2 we present a short review of Lie group representation of rigid body kinematics and the exponential map on SE(3). Sect. 3 is devoted to the fundamentals of the rigid body formulation with non-redundant unified local velocity coordinates as well as to the derivation of the equations of motion and their generalized forces. In Sect. 4, we show the relation of the proposed formulation to the Newton-Euler equations and present a further analysis of the proposed rigid body formulation. The proper time integration of the equations of motion and the non-redundant unified local velocity coordinates is shown in Sect. 5. Lastly, the proposed formulation is validated through a numerical example in Sect. 6.

2 Lie group framework

Since the six unified velocity coordinates are defined in a body-fixed frame, standard integrators such as the generalized-α method [18] can not be used to obtain (drift-free) information about the current position and orientation of a rigid body via direct numerical integration. However, Lie group time integrators based on the exponential map such as e.g. the ones presented by Crouch and Grossmann [14], Munthe-Kaas [15, 16], Brüls and Cardona [17], Terze et al. [20] or by Arnold et al. [8] can be used in an adapted form, which is why we introduce in this section the Lie group representation of rigid body kinematics on the special Euclidean group SE(3).

In the following, capital boldface letters represent matrices and lower case letters represent vectors. The horizontal line over vectors $\bar{\mathbf{a}}$ means that the vector is rigidly attached to the body and the coordinates are given in the (local) body frame. The components of a vector are written in non-bold letters. If a horizontal line is drawn above the components of a vector, this means that these are the coordinates of the vector represented in the body-fixed frame. The same convention is used for matrices and their components.

The position of a rigid body in a reference frame I is represented by a vector $\mathbf{a} \in \mathbb{R}^3$. To obtain a singularity-free representation of the orientation of the rigid body with respect to a reference frame, for example either a rotation matrix

$$\mathbf{R} \in \mathrm{SO}(3) := \left\{ \mathbf{R} \in \mathbb{R}^{3\times 3} \mid \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}_3,\ \det(\mathbf{R}) = +1 \right\} \tag{1}$$

or Euler parameters are used as degrees of freedom [2]. Rigid body transformations are represented by frame transformations, which belong to the special Euclidean group SE(3) [13]. The action of a rigid body transformation on a body-fixed frame describes how the body-fixed frame rotates and displaces with respect to a inertial frame [21]. Rigid body transformations are such that the position vector $\mathbf{x}_p \in \mathbb{R}^3$ of any point P in the current configuration on the rigid body is expressed by the affine mapping

$$\mathbf{x}_p = \mathbf{x} + \mathbf{y}_p = \mathbf{x} + \mathbf{R}\bar{\mathbf{y}}_p \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^3$ represents the current position vector of a reference point, which was located at the point $\boldsymbol{o}$ in the reference configuration [13]. This is illustrated in Fig. 2. The vector $\mathbf{y}_p = \mathbf{R}\bar{\mathbf{y}}_p$ represents the position of an arbitrary point P of the body with respect to the fixed reference frame I. Whereas the vector $\bar{\mathbf{y}}_p \in \mathbb{R}^3$ represents the position of the point P in the body-fixed frame. The rotation matrix $\mathbf{R}$ maps the vector $\bar{\mathbf{y}}_p$ from the body-fixed frame into the reference frame and defines the orientation of the body-fixed frame with respect to the reference frame I. A rigid body transformation (2) can conveniently be represented in matrix form using the homogeneous representation of vectors as

$$\begin{bmatrix} \mathbf{x}_p \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{x} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{y}}_p \\ 1 \end{bmatrix} = \mathbf{H}(\mathbf{R}, \mathbf{x}) \begin{bmatrix} \bar{\mathbf{y}}_p \\ 1 \end{bmatrix}. \tag{3}$$

The set of 4×4-matrices, which are formed by the semi-direct product $\mathrm{SO}(3) \ltimes \mathbb{R}^3$,

$$\mathbf{H} = \mathbf{H}(\mathbf{R}, \mathbf{x}) := \left\{ \mathbf{H} \in \mathbb{R}^{4\times 4} \mid \mathbf{H} \in \mathrm{SO}(3) \ltimes \mathbb{R}^3 \right\}, \quad \mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{x} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \tag{4}$$

where $\mathbf{R} \in \mathrm{SO}(3)$ and $\mathbf{x} \in \mathbb{R}^3$ forms together with the matrix product (composition operation) a matrix Lie group, which is called the special Euclidean group SE(3) [22, 13]. The velocity vector $\dot{\mathbf{x}}_p \in \mathbb{R}^3$ of any point P on the rigid

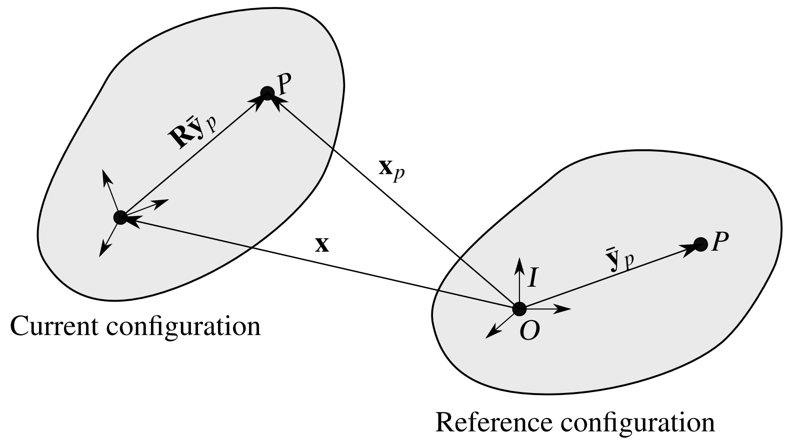


Fig. 2: Rigid body transformation.

body can be expressed in the current frame as linear relation between $\dot{\mathbf{H}}$ and $\begin{bmatrix}\bar{\mathbf{y}}_p^{\mathrm{T}} & 1\end{bmatrix}^{\mathrm{T}}$ in the form

$$\begin{bmatrix}\dot{\mathbf{x}}_p \\ 0\end{bmatrix} = \dot{\mathbf{H}}\begin{bmatrix}\bar{\mathbf{y}}_p \\ 1\end{bmatrix}. \tag{5}$$

The time derivative $\dot{\mathbf{H}}$ reads

$$\dot{\mathbf{H}} = \begin{bmatrix}\mathbf{R}\widetilde{\overline{\boldsymbol{\Omega}}} & \mathbf{R}\overline{\mathbf{U}} \\ \mathbf{0}_{1\times3} & 0\end{bmatrix} = \begin{bmatrix}\mathbf{R} & \mathbf{x} \\ \mathbf{0}_{1\times3} & 1\end{bmatrix}\begin{bmatrix}\widetilde{\overline{\boldsymbol{\Omega}}} & \overline{\mathbf{U}} \\ \mathbf{0}_{1\times3} & 0\end{bmatrix} = \mathbf{H}\widetilde{\overline{\mathbf{v}}}_{\mathrm{L}}. \tag{6}$$

The 4×4-matrix

$$\widetilde{\overline{\mathbf{v}}}_{\mathrm{L}} = \begin{bmatrix}\widetilde{\overline{\boldsymbol{\Omega}}} & \overline{\mathbf{U}} \\ \mathbf{0}_{1\times3} & 0\end{bmatrix} \tag{7}$$

denotes the Lie algebra $\mathfrak{se}(3)$ corresponding to an element of SE(3) with $\mathfrak{so}(3)$ defined by

$$\widetilde{\overline{\boldsymbol{\Omega}}} \in \mathfrak{so}(3) := \left\{\widetilde{\overline{\boldsymbol{\Omega}}} \in \mathbb{R}^{3\times3} \mid \widetilde{\overline{\boldsymbol{\Omega}}} + \widetilde{\overline{\boldsymbol{\Omega}}}^{\mathrm{T}} = \mathbf{0}_{3\times3}\right\}, \quad \widetilde{\overline{\boldsymbol{\Omega}}} = \begin{bmatrix}0 & -\overline{\Omega}_3 & \overline{\Omega}_2 \\ \overline{\Omega}_3 & 0 & -\overline{\Omega}_1 \\ -\overline{\Omega}_2 & \overline{\Omega}_1 & 0\end{bmatrix}, \quad \widetilde{\overline{\boldsymbol{\Omega}}} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} \tag{8}$$

being the Lie algebra corresponding to SO(3) and $\dot{\mathbf{x}} = \mathbf{R}\overline{\mathbf{U}}$ with $\overline{\mathbf{U}} \in \mathbb{R}^3$. A Lie algebra $\mathfrak{g}$ is isomorphic to $\mathbb{R}^k$ trough an invertible linear map

$$\widetilde{(\bullet)} : \mathbb{R}^k \to \mathfrak{g}, \qquad \mathbf{q} \mapsto \widetilde{\mathbf{q}} \tag{9}$$

with k being the dimension of the manifold [13]. Therefore, the inverse mapping (9) applied to the Lie algebra (7) maps the 4×4-matrix $\widetilde{\overline{\mathbf{v}}}_{\mathrm{L}}$ to a six-dimensional vector

$$\overline{\mathbf{v}}_{\mathrm{L}} = \begin{bmatrix}\overline{\mathbf{U}}^{\mathrm{T}} & \overline{\boldsymbol{\Omega}}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} \tag{10}$$

and vice versa. In the vector representation $\overline{\mathbf{v}}_{\mathrm{L}}$ of the Lie algebra $\widetilde{\overline{\mathbf{v}}}_{\mathrm{L}}$, the sub-vector $\overline{\mathbf{U}} = \begin{bmatrix}\overline{\mathrm{U}}_1 & \overline{\mathrm{U}}_2 & \overline{\mathrm{U}}_3\end{bmatrix}^{\mathrm{T}}$ represents the translational velocity in a body-fixed frame and the sub-vector $\overline{\boldsymbol{\Omega}} = \begin{bmatrix}\overline{\Omega}_1 & \overline{\Omega}_2 & \overline{\Omega}_3\end{bmatrix}^{\mathrm{T}}$ represents the angular velocity vector in a body-fixed frame [12, 8]. The differential Eq. (6) is a first order differential equation on a matrix Lie group, which produces a relation between $\mathbf{H}$ and $\dot{\mathbf{H}}$ [22, 23]. This differential equation can be solved in every time step with the exponential map $\exp_{\mathrm{SE(3)}}$ on SE(3)

$$\dot{\mathbf{n}}(\mathrm{t}+\mathrm{h}) = \mathbf{T}^{-1}_{\mathrm{SE(3)}}(\mathbf{n}(\mathrm{t}))\,\overline{\mathbf{v}}_{\mathrm{L}}(\mathrm{t}+\mathrm{h}), \qquad \mathbf{n}(0) = \mathbf{0}_{6\times1}, \tag{11}$$

$$\mathbf{H}(\mathrm{t}+\mathrm{h}) = \mathbf{H}(\mathrm{t})\,\exp_{\mathrm{SE(3)}}(\mathbf{n}(\mathrm{t}+\mathrm{h})), \qquad \mathbf{H}(0) = \mathbf{H}_0, \tag{12}$$

after the time integration, the incremental motion vector $\mathbf{n} = \begin{bmatrix} \mathrm{n}_1 & \mathrm{n}_2 & \mathrm{n}_3 & \mathrm{n}_4 & \mathrm{n}_5 & \mathrm{n}_6 \end{bmatrix}^{\mathrm{T}}$ with its sub-vectors $\mathbf{n}_{\mathrm{T}} = \begin{bmatrix} \mathrm{n}_1 & \mathrm{n}_2 & \mathrm{n}_3 \end{bmatrix}^{\mathrm{T}}$ and $\mathbf{n}_{\mathrm{R}} = \begin{bmatrix} \mathrm{n}_4 & \mathrm{n}_5 & \mathrm{n}_6 \end{bmatrix}^{\mathrm{T}}$ is obtained from Eq. (11) at the corresponding time step [22, 12, 13]. To point this out, we have written the Eqs. (11-12) in discretized form for a specific time step $\mathrm{t}+\mathrm{h}$, where h is a small step forward in time. In Eq. (11), $\mathbf{T}^{-1}_{\mathrm{SE(3)}}$ denotes the inverse tangent operator on SE(3) [12]. The exponential map and the inverse tangent operator on SE(3) can be written in a closed form

$$\exp_{\mathrm{SE(3)}}(\mathbf{n}) = \begin{bmatrix} \exp_{\mathrm{SO(3)}}(\mathbf{n}_{\mathrm{R}}) & \mathbf{T}^{\mathrm{T}}_{\mathrm{SO(3)}}(\mathbf{n}_{\mathrm{R}})\mathbf{n}_{\mathrm{T}} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}, \qquad \mathbf{T}^{-1}_{\mathrm{SE(3)}}(\mathbf{n}) = \begin{bmatrix} \mathbf{T}^{-1}_{\mathrm{SO(3)}}(\mathbf{n}_{\mathrm{R}}) & \mathbf{T}_{\mathrm{n_T n_R}-}(\mathbf{n}_{\mathrm{T}}, \mathbf{n}_{\mathrm{R}}) \\ \mathbf{0}_{3\times 3} & \mathbf{T}^{-1}_{\mathrm{SO(3)}}(\widetilde{\mathbf{n}}_{\mathrm{R}}) \end{bmatrix} \tag{13}$$

where $\exp_{\mathrm{SO(3)}}$ is the exponential map and $\mathbf{T}_{\mathrm{SO(3)}}$ the tangent operator on the special orthogonal group SO(3) [12, 8], which are given by

$$\exp_{\mathrm{SO(3)}}(\mathbf{n}_{\mathrm{R}}) = \mathbf{I}_{3\times 3} + \frac{\sin(\|\mathbf{n}_{\mathrm{R}}\|)}{\|\mathbf{n}_{\mathrm{R}}\|}\widetilde{\mathbf{n}}_{\mathrm{R}} + \frac{1-\cos(\|\mathbf{n}_{\mathrm{R}}\|)}{\|\mathbf{n}_{\mathrm{R}}\|^2}\widetilde{\mathbf{n}}^2_{\mathrm{R}}, \tag{14}$$

$$\mathbf{T}_{\mathrm{SO(3)}}(\mathbf{n}_{\mathrm{R}}) = \mathbf{I}_{3\times 3} - \frac{1-\cos(\|\mathbf{n}_{\mathrm{R}}\|)}{\|\mathbf{n}_{\mathrm{R}}\|^2}\widetilde{\mathbf{n}}_{\mathrm{R}} + \frac{\|\mathbf{n}_{\mathrm{R}}\| - \sin(\|\mathbf{n}_{\mathrm{R}}\|)}{\|\mathbf{n}_{\mathrm{R}}\|^3}\widetilde{\mathbf{n}}^2_{\mathrm{R}}, \tag{15}$$

$$\mathbf{T}^{-1}_{\mathrm{SO(3)}}(\mathbf{n}_{\mathrm{R}}) = \mathbf{I}_{3\times 3} + \frac{1}{2}\widetilde{\mathbf{n}}_{\mathrm{R}} + \frac{\frac{1}{2}(2-\cot(\frac{\|\mathbf{n}_{\mathrm{R}}\|}{2}))}{\|\mathbf{n}_{\mathrm{R}}\|^2}\widetilde{\mathbf{n}}^2_{\mathrm{R}}, \tag{16}$$

$$\mathbf{T}_{\mathrm{n_T n_R}-}(\mathbf{n}_{\mathrm{T}}, \mathbf{n}_{\mathrm{R}}) = \frac{1}{2}\widetilde{\mathbf{n}}_{\mathrm{T}} + \frac{\frac{1}{2}(2-\cot(\frac{\|\mathbf{n}_{\mathrm{R}}\|}{2}))}{\|\mathbf{n}_{\mathrm{R}}\|^2}\lceil \widetilde{\mathbf{n}}_{\mathrm{T}}, \widetilde{\mathbf{n}}_{\mathrm{R}} \rceil + \frac{\frac{1}{2}\left(\frac{\|\mathbf{n}_{\mathrm{R}}\|^2}{1-\cos(\|\mathbf{n}_{\mathrm{R}}\|)} + \cot(\frac{\|\mathbf{n}_{\mathrm{R}}\|}{2})\right)}{\|\mathbf{n}_{\mathrm{R}}\|^4}\left(\mathbf{n}^{\mathrm{T}}_{\mathrm{R}}\mathbf{n}_{\mathrm{T}}\right)\widetilde{\mathbf{n}}^2_{\mathrm{R}}, \tag{17}$$

using the the Lie bracket $\lceil , \rceil$, see [15]. The matrices $\widetilde{\mathbf{n}}_{\mathrm{R}}$ and $\widetilde{\mathbf{n}}_{\mathrm{T}}$ are skew symmetric matrices of the form (8), which are formed from the vectors $\mathbf{n}_{\mathrm{R}}$ and $\mathbf{n}_{\mathrm{T}}$.

3 Spatial rigid body motion using non-redundant unified local velocity coordinates

In many approaches [7, 6, 2], the kinematics of the rigid body and its equations of motion are divided into a translational part of motion and a rotational part of motion. This subdivision of the general form of motion at velocity level is currently also used within the Lie group framework, see the previous section or e.g., [12, 8, 13]. Since a unification of the velocity coordinates within the Lie group description of the rigid body motion could be advantageous, we show in the following how the general state of motion of a rigid body at velocity and acceleration level can be unified and clearly described using non-redundant local velocity coordinates.

3.1 Non-redundant unified local velocity coordinates

The unification of the local translational velocity coordinates $(\overline{\mathrm{U}}_1, \overline{\mathrm{U}}_2, \overline{\mathrm{U}}_3)$ and the local angular velocity coordinates $(\overline{\Omega}_1, \overline{\Omega}_2, \overline{\Omega}_3)$ to six local velocity coordinates $(\overline{\mathrm{w}}_1, \overline{\mathrm{w}}_2, \overline{\mathrm{w}}_3, \overline{\mathrm{w}}_4, \overline{\mathrm{w}}_5, \overline{\mathrm{w}}_6)$ is achieved, by projecting the local translational velocity coordinates of six selected points on the rigid body along six selected directions. The scalar product

$$\overline{\mathrm{w}}_i := \bar{\mathbf{b}}^{\mathrm{T}}_i \bar{\mathbf{v}}_i(\bar{\mathbf{y}}_i) \tag{18}$$

generates the i-th unified local velocity coordinate $\overline{\mathrm{w}}_i \in \mathbb{R}$. The i-th local translational velocity vector of the i-th point P_i selected for the projection is indicated by the vector $\bar{\mathbf{v}}_i \in \mathbb{R}^3$. The local position of point P_i on the rigid body is marked with vector $\bar{\mathbf{y}}_i \in \mathbb{R}^3$. The vector $\bar{\mathbf{b}}_i \in \mathbb{R}^3$ represents the i-th direction vector selected for the projection. The general projection idea is depicted in Fig. 3. The term unified here means, that when describing the velocity state of a rigid body with the coordinates $\overline{\mathrm{w}}$, no distinction is made between a translational motion and a rotational motion. The six coordinates $\overline{\mathrm{w}}_i$ with $i = 1 \ldots 6$ are non-redundant velocity coordinates, since six velocity coordinates clearly describe the Lie algebra (7) of a rigid body. The set of projection points P and direction vectors $\bar{\mathbf{b}}$ to generate

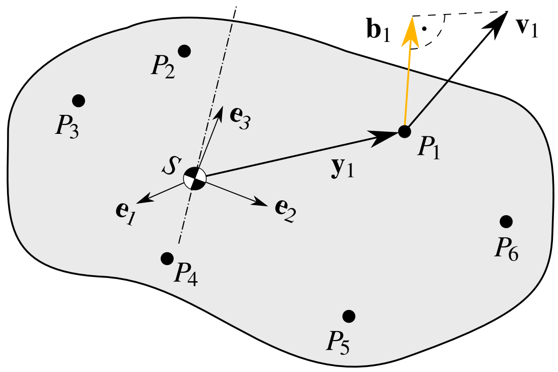


Fig. 3: Basic projection idea to obtain unified local velocity coordinates. S denotes the center of mass of the rigid body. The six points, used for the velocity projection are denoted with $p_1 \ldots p_6$. Note that $\mathbf{b}_1$, $\mathbf{v}_1$, $\mathbf{y}_1$ and $\mathbf{e}_i$ are the global representations of their body-fixed counterparts $\bar{\mathbf{b}}_1$, $\bar{\mathbf{v}}_1$, $\bar{\mathbf{y}}_1$ and $\bar{\mathbf{e}}_i$.

the six unified velocity coordinates $\overline{\mathrm{w}}$ is limited by the condition that the velocity coordinates $\overline{\mathrm{w}}$ must be clearly mappable to the vector form of the Lie algebra $\bar{\mathbf{v}}_{\mathrm{L}}$. This restriction to the set of permitted projection points and direction vectors is necessary, as otherwise the Lie algebra $\widetilde{\bar{\mathbf{v}}}_{\mathrm{L}}$ cannot be clearly reconstructed from the unified local velocity coordinates. We will discuss this condition in more detail in section 4.1. Therefore, theoretically many projection approaches could be applied. For example a projection towards the lateral surfaces of a hexahedron or towards the edges of a tetrahedron as shown in Fig. 4. In order to be able to examine the description of the rigid body motion with unified local velocity coordinates more closely, we will limit ourselves for simplicity's sake to the velocity projection along the side edges of a hexahedron with the side lengths ($2\mathrm{l}_1$, $2\mathrm{l}_2$, $2\mathrm{l}_3$) as depicted Fig. 4 a. The position vectors $\bar{\mathbf{y}}_i$ necessary for generating six unified velocity coordinates belonging to six points located at the centers of the side faces of the hexahedron and the six directional vectors $\bar{\mathbf{b}}_i$,

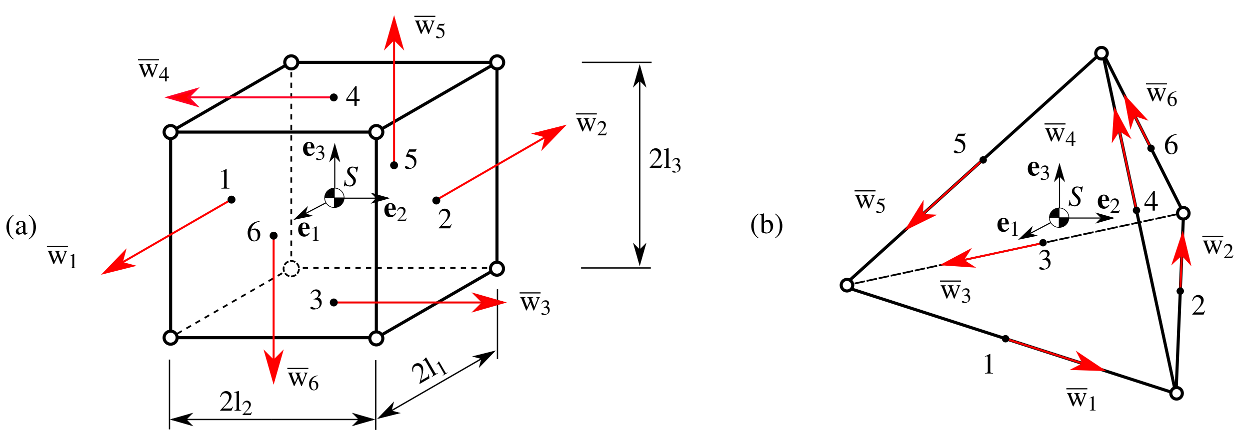


Fig. 4: Basic geometries for unified local velocity coordinates. S denotes the center of mass of the rigid body. The six points, used for the velocity projection are denoted with $1 \ldots 6$. The corresponding six unified velocities are denoted by $\overline{\mathrm{w}}_1 \ldots \overline{\mathrm{w}}_6$. Fig. a) shows the hexahedral case, Fig b) shows the tetrahedral case.

$$\bar{\mathbf{b}}_1 = \begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & 0\end{bmatrix}^{\mathrm{T}}, \qquad \bar{\mathbf{y}}_1 = \begin{bmatrix}0 & -\mathrm{l}_2 & 0\end{bmatrix}^{\mathrm{T}}, \qquad \overline{\mathrm{w}}_1 := \begin{bmatrix}\bar{\mathbf{b}}_1^{\mathrm{T}} & \left(\tilde{\bar{\mathbf{y}}}_1\bar{\mathbf{b}}_1\right)^{\mathrm{T}}\end{bmatrix}\bar{\mathbf{v}}_{\mathrm{L}}, \tag{19}$$

$$\bar{\mathbf{b}}_2 = \begin{bmatrix}\frac{-1}{\sqrt{2}} & 0 & 0\end{bmatrix}^{\mathrm{T}}, \qquad \bar{\mathbf{y}}_2 = \begin{bmatrix}0 & \mathrm{l}_2 & 0\end{bmatrix}^{\mathrm{T}}, \qquad \overline{\mathrm{w}}_2 := \begin{bmatrix}\bar{\mathbf{b}}_2^{\mathrm{T}} & \left(\tilde{\bar{\mathbf{y}}}_2\bar{\mathbf{b}}_2\right)^{\mathrm{T}}\end{bmatrix}\bar{\mathbf{v}}_{\mathrm{L}}, \tag{20}$$

$$\bar{\mathbf{b}}_3 = \begin{bmatrix}0 & \frac{1}{\sqrt{2}} & 0\end{bmatrix}^{\mathrm{T}}, \qquad \bar{\mathbf{y}}_3 = \begin{bmatrix}0 & 0 & -\mathrm{l}_3\end{bmatrix}^{\mathrm{T}}, \qquad \overline{\mathrm{w}}_3 := \begin{bmatrix}\bar{\mathbf{b}}_3^{\mathrm{T}} & \left(\tilde{\bar{\mathbf{y}}}_3\bar{\mathbf{b}}_3\right)^{\mathrm{T}}\end{bmatrix}\bar{\mathbf{v}}_{\mathrm{L}}, \tag{21}$$

$$\bar{\mathbf{b}}_4 = \begin{bmatrix}0 & \frac{-1}{\sqrt{2}} & 0\end{bmatrix}^{\mathrm{T}}, \qquad \bar{\mathbf{y}}_4 = \begin{bmatrix}0 & 0 & \mathrm{l}_3\end{bmatrix}^{\mathrm{T}}, \qquad \overline{\mathrm{w}}_4 := \begin{bmatrix}\bar{\mathbf{b}}_4^{\mathrm{T}} & \left(\tilde{\bar{\mathbf{y}}}_4\bar{\mathbf{b}}_4\right)^{\mathrm{T}}\end{bmatrix}\bar{\mathbf{v}}_{\mathrm{L}}, \tag{22}$$

$$\bar{\mathbf{b}}_5 = \begin{bmatrix}0 & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}^{\mathrm{T}}, \qquad \bar{\mathbf{y}}_5 = \begin{bmatrix}-\mathrm{l}_1 & 0 & 0\end{bmatrix}^{\mathrm{T}}, \qquad \overline{\mathrm{w}}_5 := \begin{bmatrix}\bar{\mathbf{b}}_5^{\mathrm{T}} & \left(\tilde{\bar{\mathbf{y}}}_5\bar{\mathbf{b}}_5\right)^{\mathrm{T}}\end{bmatrix}\bar{\mathbf{v}}_{\mathrm{L}}, \tag{23}$$

$$\bar{\mathbf{b}}_6 = \begin{bmatrix}0 & 0 & \frac{-1}{\sqrt{2}}\end{bmatrix}^{\mathrm{T}}, \qquad \bar{\mathbf{y}}_6 = \begin{bmatrix}\mathrm{l}_1 & 0 & 0\end{bmatrix}^{\mathrm{T}}, \qquad \overline{\mathrm{w}}_6 := \begin{bmatrix}\bar{\mathbf{b}}_6^{\mathrm{T}} & \left(\tilde{\bar{\mathbf{y}}}_6\bar{\mathbf{b}}_6\right)^{\mathrm{T}}\end{bmatrix}\bar{\mathbf{v}}_{\mathrm{L}}, \tag{24}$$

clearly map a general velocity state, which is given by the vector representation $\bar{\mathbf{v}}_{\mathrm{L}}$ of the Lie algebra $\tilde{\tilde{\bar{\mathbf{v}}}}_{\mathrm{L}}$ to six unified velocity coordinates

$$\overline{\mathbf{w}} := \begin{bmatrix}\overline{\mathrm{w}}_1 & \overline{\mathrm{w}}_2 & \overline{\mathrm{w}}_3 & \overline{\mathrm{w}}_4 & \overline{\mathrm{w}}_5 & \overline{\mathrm{w}}_6\end{bmatrix}^{\mathrm{T}}. \tag{25}$$

In section 4.1 we will show why exactly this choice of direction and position vectors, including the factor $1/\sqrt{2}$, is advantageous and why they can clearly map the velocity coordinates $\overline{\mathrm{w}}$ to the Lie algebra.

3.1.1 Velocity field

The local translational velocity vector $\bar{\mathbf{v}}(\bar{\mathbf{y}}_p) = \begin{bmatrix}\bar{\mathrm{v}}_1 & \bar{\mathrm{v}}_2 & \bar{\mathrm{v}}_3\end{bmatrix}^{\mathrm{T}}$ of each point P on the rigid body, whose position on the rigid body is given by the local vector $\bar{\mathbf{y}}_p$, can be calculated from the six local unified velocity coordinates $\overline{\mathbf{w}}$ using the interpolation

$$\bar{\mathbf{v}}(\bar{\mathbf{y}}_p) = \overline{\mathbf{N}}(\bar{\mathbf{y}}_p)\overline{\mathbf{w}} \tag{26}$$

where the interpolation matrix $\overline{\mathbf{N}}(\bar{\mathbf{y}}_p)$

$$\overline{\mathbf{N}}(\bar{\mathbf{y}}_p) = \frac{1}{\sqrt{2}}\begin{bmatrix}1-\frac{\bar{\mathrm{y}}_2}{\mathrm{l}_2} & -1-\frac{\bar{\mathrm{y}}_2}{\mathrm{l}_2} & 0 & 0 & \frac{\bar{\mathrm{y}}_3}{\mathrm{l}_1} & \frac{\bar{\mathrm{y}}_3}{\mathrm{l}_1} \\ \frac{\bar{\mathrm{y}}_1}{\mathrm{l}_2} & \frac{\bar{\mathrm{y}}_1}{\mathrm{l}_2} & 1-\frac{\bar{\mathrm{y}}_3}{\mathrm{l}_3} & -1-\frac{\bar{\mathrm{y}}_3}{\mathrm{l}_3} & 0 & 0 \\ 0 & 0 & \frac{\bar{\mathrm{y}}_2}{\mathrm{l}_3} & \frac{\bar{\mathrm{y}}_2}{\mathrm{l}_3} & 1-\frac{\bar{\mathrm{y}}_1}{\mathrm{l}_1} & -1-\frac{\bar{\mathrm{y}}_1}{\mathrm{l}_1}\end{bmatrix} = \begin{bmatrix}\overline{\mathbf{N}}_1^{1\times 6}(\bar{\mathbf{y}}_p) \\ \overline{\mathbf{N}}_2^{1\times 6}(\bar{\mathbf{y}}_p) \\ \overline{\mathbf{N}}_3^{1\times 6}(\bar{\mathbf{y}}_p)\end{bmatrix} \tag{27}$$

is given in terms of the hexahedron dimensions and the components of the local position vector $\bar{\mathbf{y}}_p = \begin{bmatrix}\bar{\mathrm{y}}_1 & \bar{\mathrm{y}}_2 & \bar{\mathrm{y}}_3\end{bmatrix}^{\mathrm{T}}$.

To show that the velocity field of the rigid body is correctly described by Eq. (26), we show the derivation of the first line of Eq. (26). From a kinematic point of view, the velocity component $\bar{\mathrm{v}}_1$ is composed of three translational velocity components, which result from different forms of motion of the rigid body. Namely a translational velocity component $\bar{\mathrm{v}}_1^{\mathrm{transl}}$, which occurs due to a pure translational motion of the rigid body towards the local basis vector $\mathbf{e}_1$. A velocity component $\bar{\mathrm{v}}_1^{\mathrm{rot}_{\mathbf{e}_2}}$, which results from a pure rotary motion of the rigid body around the axis of rotation whose direction is given by the local basis vector $\mathbf{e}_2$. As well as a velocity component $\bar{\mathrm{v}}_1^{\mathrm{rot}_{\mathbf{e}_3}}$, which results from a rotational motion of the rigid body around the axis of rotation whose direction is given by the local basis vector $\mathbf{e}_3$. From the definition (18) of the six unified velocity coordinates $\overline{\mathrm{w}}$ it follows that $\overline{\mathrm{w}}_1 = \frac{1}{\sqrt{2}}\bar{\mathrm{v}}_{\mathrm{trans}}^{\mathrm{e1}}$ and $\overline{\mathrm{w}}_2 = \frac{-1}{\sqrt{2}}\bar{\mathrm{v}}_{\mathrm{trans}}^{\mathrm{e1}}$. By the projection (26) the velocity vector $\bar{\mathbf{v}}(\bar{\mathbf{y}})$ is not only projected in a direction given by the direction vectors b, but also scaled at the same time. This is illustrated in Fig. 5 a.

In the case of a pure translational motion, the velocity of any reference point $\bar{\mathrm{v}}_{\mathrm{Ref}}$ on the rigid body therefore corresponds by

$$\bar{\mathrm{v}}_{\mathrm{Ref}} = \overline{\mathrm{w}}_1 - \overline{\mathrm{w}}_2 = \sqrt{2}\,\bar{\mathrm{v}}_1^{\mathrm{transl}} \quad \Rightarrow \quad \bar{\mathrm{v}}_1^{\mathrm{transl}} = \frac{1}{\sqrt{2}}\left(\overline{\mathrm{w}}_1 - \overline{\mathrm{w}}_2\right). \tag{28}$$

The velocity component $\bar{\mathrm{v}}_1^{\mathrm{rot}_{\mathbf{e}_3}}$ follows from linear interpolation

$$\bar{\mathrm{v}}_1^{\mathrm{rot}_{\mathbf{e}_3}}(\bar{\mathrm{y}}_2) = -\frac{1}{\sqrt{2}}\left(\frac{\overline{\mathrm{w}}_1}{\mathrm{l}_2} + \frac{\overline{\mathrm{w}}_2}{\mathrm{l}_2}\right)\bar{\mathrm{y}}_2 \tag{29}$$

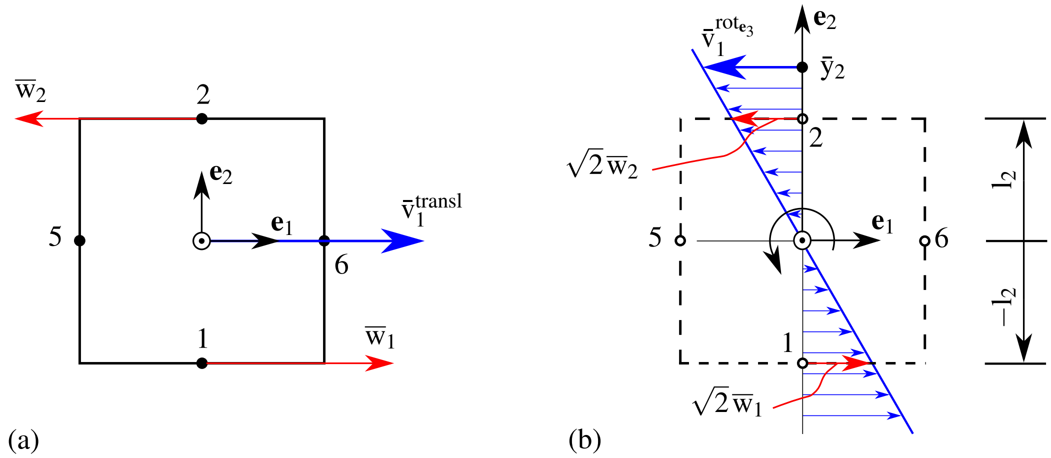


Fig. 5: Representation of the motion of the cross-section spanned by the transformation points (1,2,5,6), see also Fig. 4 a.

according to Fig. 5 b. The negative slope in $\bar{\mathrm{v}}_1^{\mathrm{rot}_{\mathbf{e}_3}}$ follows from the assumed mathematically positive sense of rotation. In the same sense the velocity component $\bar{\mathrm{v}}_1^{\mathrm{rot}_{\mathbf{e}_2}}$ follows from linear interpolation

$$\bar{\mathrm{v}}_1^{\mathrm{rot}_{\mathbf{e}_2}}(\bar{\mathrm{y}}_3) = \frac{1}{\sqrt{2}}\left(\frac{\overline{\mathrm{w}}_5}{\mathrm{l}_1} + \frac{\overline{\mathrm{w}}_6}{\mathrm{l}_1}\right)\bar{\mathrm{y}}_3 \tag{30}$$

according to Fig. 6 b. Summing up the terms $\bar{\mathrm{v}}_1^{\mathrm{transl}}$, $\bar{\mathrm{v}}_1^{\mathrm{rot}_{\mathbf{e}_2}}$ and $\bar{\mathrm{v}}_1^{\mathrm{rot}_{\mathbf{e}_3}}$ results in the first line of Eq. (26)

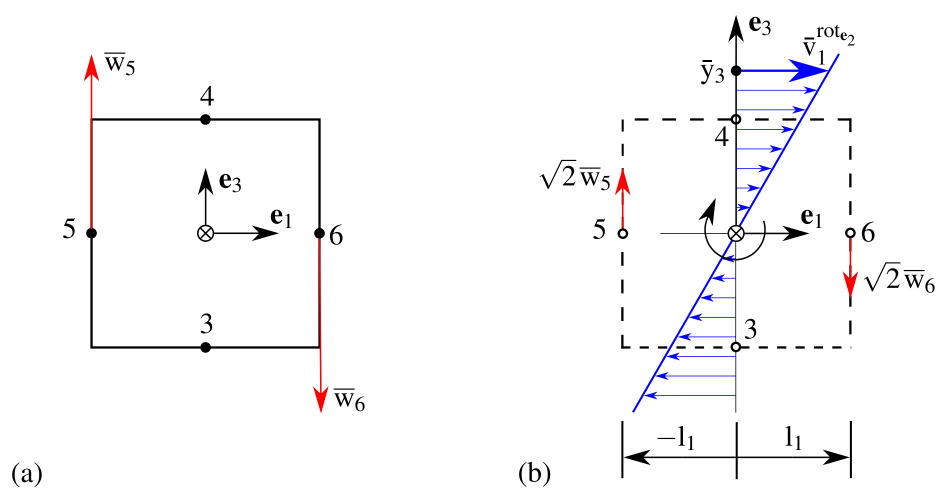


Fig. 6: Representation of the motion of the cross-section spanned by the transformation points (3,4,5,6), see also Fig. 4 a.

$$\bar{\mathrm{v}}_1 = \bar{\mathrm{v}}_1^{\mathrm{transl}} + \bar{\mathrm{v}}_1^{\mathrm{rot}_{\mathbf{e}_2}} + \bar{\mathrm{v}}_1^{\mathrm{rot}_{\mathbf{e}_3}} = \frac{1}{\sqrt{2}}(\overline{\mathrm{w}}_1 - \overline{\mathrm{w}}_2) + \frac{1}{\sqrt{2}}\left(\frac{\overline{\mathrm{w}}_5}{\mathrm{l}_1} + \frac{\overline{\mathrm{w}}_6}{\mathrm{l}_1}\right)\bar{\mathrm{y}}_3 - \frac{1}{\sqrt{2}}\left(\frac{\overline{\mathrm{w}}_1}{\mathrm{l}_2} + \frac{\overline{\mathrm{w}}_2}{\mathrm{l}_2}\right)\bar{\mathrm{y}}_2. \tag{31}$$

If Eq. (31) is formulated as a scalar product of a 1×6-vector with the vector $\overline{\mathbf{w}}$, the 1×6-vector appears to be the first line $\overline{\mathbf{N}}_1^{1\times 6}(\bar{\mathbf{y}}_p)$ of the interpolation matrix shown in Eq. (27). Of course, these relations would also follow from basic kinematic formulas using the angular velocity vector.

3.2 Representation of accelerations

A unification of the kinematic description of the rigid body motion at velocity level is possible with the help of the six local velocity coordinates $\overline{\mathrm{w}}_i$ as shown in the previous section. A unified kinematic representation of the rigid body motion on acceleration level can also be found based on the time derivatives of the unified local velocity coordinates $\overline{\mathrm{w}}_i$. The velocity field $\overline{\mathbf{v}}$ in Eq. (26) can be written in coordinate representation

$$\mathbf{v}(\bar{\mathbf{y}}_p) = \bar{\mathrm{v}}_1\mathbf{e}_1 + \bar{\mathrm{v}}_2\mathbf{e}_2 + \bar{\mathrm{v}}_3\mathbf{e}_3 = \left(\overline{\mathbf{N}}_1(\bar{\mathbf{y}}_p)\overline{\mathbf{w}}\right)\mathbf{e}_1 + \left(\overline{\mathbf{N}}_2(\bar{\mathbf{y}}_p)\overline{\mathbf{w}}\right)\mathbf{e}_2 + \left(\overline{\mathbf{N}}_3(\bar{\mathbf{y}}_p)\overline{\mathbf{w}}\right)\mathbf{e}_3, \tag{32}$$

where $(\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3)$ represent the three body-fixed local orthogonal basis vectors, with their (time dependent) coordinates represented in the global frame. The acceleration vector of any point P in the current configuration can be determined from the time derivative of Eq. (32). Therefore, the acceleration vector follows as

$$\begin{aligned}\dot{\mathbf{v}}(\bar{\mathbf{y}}_p) = {}& \left(\overline{\mathbf{N}}_1(\bar{\mathbf{y}}_p)\dot{\overline{\mathbf{w}}}\right)\mathbf{e}_1 + \left(\overline{\mathbf{N}}_2(\bar{\mathbf{y}}_p)\dot{\overline{\mathbf{w}}}\right)\mathbf{e}_2 + \left(\overline{\mathbf{N}}_3(\bar{\mathbf{y}}_p)\dot{\overline{\mathbf{w}}}\right)\mathbf{e}_3 + \\ & + \left(\overline{\mathbf{N}}_1(\bar{\mathbf{y}}_p)\overline{\mathbf{w}}\right)\dot{\mathbf{e}}_1 + \left(\overline{\mathbf{N}}_2(\bar{\mathbf{y}}_p)\overline{\mathbf{w}}\right)\dot{\mathbf{e}}_2 + \left(\overline{\mathbf{N}}_3(\bar{\mathbf{y}}_p)\overline{\mathbf{w}}\right)\dot{\mathbf{e}}_3,\end{aligned} \tag{33}$$

where $(\dot{\mathbf{e}}_1,\ \dot{\mathbf{e}}_2,\ \dot{\mathbf{e}}_3)$ represent the time derivatives of the three body-fixed local orthogonal basis vectors. Formulated in matrix-vector notation, the acceleration vector (33) takes the form

$$\dot{\mathbf{v}}(\bar{\mathbf{y}}_p) = [\mathbf{e}_1\ \ \mathbf{e}_2\ \ \mathbf{e}_3]\left[\overline{\mathbf{N}}_1^{\mathrm{T}}(\bar{\mathbf{y}}_p)\quad \overline{\mathbf{N}}_2^{\mathrm{T}}(\bar{\mathbf{y}}_p)\quad \overline{\mathbf{N}}_3^{\mathrm{T}}(\bar{\mathbf{y}}_p)\right]^{\mathrm{T}}\dot{\overline{\mathbf{w}}} + [\dot{\mathbf{e}}_1\ \ \dot{\mathbf{e}}_2\ \ \dot{\mathbf{e}}_3]\left[\overline{\mathbf{N}}_1^{\mathrm{T}}(\bar{\mathbf{y}}_p)\quad \overline{\mathbf{N}}_2^{\mathrm{T}}(\bar{\mathbf{y}}_p)\quad \overline{\mathbf{N}}_3^{\mathrm{T}}(\bar{\mathbf{y}}_p)\right]^{\mathrm{T}}\overline{\mathbf{w}}. \tag{34}$$

The acceleration vector transformed into the local frame reads as follows

$$\dot{\overline{\mathbf{v}}}(\bar{\mathbf{y}}_p) = [\bar{\mathbf{e}}_1\ \ \bar{\mathbf{e}}_2\ \ \bar{\mathbf{e}}_3]\left[\overline{\mathbf{N}}_1^{\mathrm{T}}(\bar{\mathbf{y}}_p)\quad \overline{\mathbf{N}}_2^{\mathrm{T}}(\bar{\mathbf{y}}_p)\quad \overline{\mathbf{N}}_3^{\mathrm{T}}(\bar{\mathbf{y}}_p)\right]^{\mathrm{T}}\dot{\overline{\mathbf{w}}} + \left[\dot{\bar{\mathbf{e}}}_1\ \ \dot{\bar{\mathbf{e}}}_2\ \ \dot{\bar{\mathbf{e}}}_3\right]\left[\overline{\mathbf{N}}_1^{\mathrm{T}}(\bar{\mathbf{y}}_p)\quad \overline{\mathbf{N}}_2^{\mathrm{T}}(\bar{\mathbf{y}}_p)\quad \overline{\mathbf{N}}_3^{\mathrm{T}}(\bar{\mathbf{y}}_p)\right]^{\mathrm{T}}\overline{\mathbf{w}}, \tag{35}$$

where $(\bar{\mathbf{e}}_1,\ \bar{\mathbf{e}}_2,\ \bar{\mathbf{e}}_3)$ represent the three body-fixed basis vectors, with their (time dependent) coordinates represented in the local frame and $(\dot{\bar{\mathbf{e}}}_1,\ \dot{\bar{\mathbf{e}}}_2,\ \dot{\bar{\mathbf{e}}}_3)$ represent the time derivatives of the three body-fixed basis vectors, with their (time dependent) coordinates represented in the local frame. In the body-fixed frame it follows that in case of a translational motion $\dot{\bar{\mathbf{e}}}_i$ vanishes. But in case of a rotational motion it follows that $\dot{\bar{\mathbf{e}}}_i \neq \mathbf{0}$. Since $\dot{\bar{\mathbf{e}}}_i$ can be interpreted as difference of two velocities, Eq. (26) is used to obtain

$$\dot{\bar{\mathbf{e}}}_i = \overline{\mathbf{v}}(\bar{\mathbf{e}}_i) - \overline{\mathbf{v}}(\mathbf{0}). \tag{36}$$

The velocity term $\overline{\mathbf{v}}(\mathbf{0})$ in Eq. (36) corresponds to a pure translational motion at $\bar{\mathbf{y}}_p = \mathbf{0}_{3\times 1}$. Eq. (36) can be evaluated using Eq. (26). By inserting the Cartesian basis vectors $\bar{\mathbf{e}}_1 = [1\ 0\ 0]^{\mathrm{T}}$, $\bar{\mathbf{e}}_2 = [0\ 1\ 0]^{\mathrm{T}}$ and $\bar{\mathbf{e}}_3 = [0\ 0\ 1]^{\mathrm{T}}$ into Eq. (36), it follows that the time derivatives of the basis vectors read

$$\dot{\bar{\mathbf{e}}}_1 = \begin{bmatrix}0 & \frac{\overline{\mathrm{w}}_1+\overline{\mathrm{w}}_2}{\sqrt{2}\mathrm{l}_2} & \frac{-\overline{\mathrm{w}}_5-\overline{\mathrm{w}}_6}{\sqrt{2}\mathrm{l}_1}\end{bmatrix}^{\mathrm{T}}, \tag{37}$$

$$\dot{\bar{\mathbf{e}}}_2 = \begin{bmatrix}\frac{-\overline{\mathrm{w}}_1-\overline{\mathrm{w}}_2}{\sqrt{2}\mathrm{l}_2} & 0 & \frac{\overline{\mathrm{w}}_3+\overline{\mathrm{w}}_4}{\sqrt{2}\mathrm{l}_3}\end{bmatrix}^{\mathrm{T}}, \tag{38}$$

$$\dot{\bar{\mathbf{e}}}_3 = \begin{bmatrix}\frac{\overline{\mathrm{w}}_5+\overline{\mathrm{w}}_6}{\sqrt{2}\mathrm{l}_1} & \frac{-\overline{\mathrm{w}}_3-\overline{\mathrm{w}}_4}{\sqrt{2}\mathrm{l}_3} & 0\end{bmatrix}^{\mathrm{T}}. \tag{39}$$

To obtain a compact vector-matrix notation of Eq. (35), the time derivatives of the basis vectors are combined in the matrix[1]

$$\dot{\overline{\mathbf{E}}} := \left[\dot{\bar{\mathbf{e}}}_1 \quad \dot{\bar{\mathbf{e}}}_2 \quad \dot{\bar{\mathbf{e}}}_3\right] = \begin{bmatrix} 0 & \frac{-\overline{\mathrm{w}}_1-\overline{\mathrm{w}}_2}{\sqrt{2}\mathrm{l}_2} & \frac{\overline{\mathrm{w}}_5+\overline{\mathrm{w}}_6}{\sqrt{2}\mathrm{l}_1} \\ \frac{\overline{\mathrm{w}}_1+\overline{\mathrm{w}}_2}{\sqrt{2}\mathrm{l}_2} & 0 & \frac{-\overline{\mathrm{w}}_3-\overline{\mathrm{w}}_4}{\sqrt{2}\mathrm{l}_3} \\ \frac{-\overline{\mathrm{w}}_5-\overline{\mathrm{w}}_6}{\sqrt{2}\mathrm{l}_1} & \frac{\overline{\mathrm{w}}_3+\overline{\mathrm{w}}_4}{\sqrt{2}\mathrm{l}_3} & 0 \end{bmatrix}. \tag{40}$$

The acceleration vector $\dot{\overline{\mathbf{v}}}(\bar{\mathbf{y}}_p)$ of any point P can be determined using

$$\dot{\overline{\mathbf{v}}}(\bar{\mathbf{y}}_p) = \overline{\mathbf{N}}(\bar{\mathbf{y}}_p)\dot{\overline{\mathbf{w}}} + \overline{\mathbf{P}}(\bar{\mathbf{y}}_p, \overline{\mathbf{w}})\overline{\mathbf{w}} \tag{41}$$

where the matrix $\overline{\mathbf{P}}(\bar{\mathbf{y}}_p, \overline{\mathbf{w}})$ is given by

$$\overline{\mathbf{P}}(\bar{\mathbf{y}}_p, \overline{\mathbf{w}}) = \dot{\overline{\mathbf{E}}}(\overline{\mathbf{w}})\overline{\mathbf{N}}(\bar{\mathbf{y}}_p) \tag{42}$$

and the time derivative of the unified coordinates reads

$$\dot{\overline{\mathbf{w}}} := \left[\dot{\overline{\mathrm{w}}}_1 \quad \dot{\overline{\mathrm{w}}}_2 \quad \dot{\overline{\mathrm{w}}}_3 \quad \dot{\overline{\mathrm{w}}}_4 \quad \dot{\overline{\mathrm{w}}}_5 \quad \dot{\overline{\mathrm{w}}}_6\right]^{\mathrm{T}}. \tag{43}$$

[1] The skew symmetric matrix $\dot{\overline{\mathbf{E}}}$ collects the vectors $\dot{\bar{\mathbf{e}}}_i$. It can be shown that in case the basis vectors correspond to the orthogonal standard basis, $\dot{\overline{\mathbf{E}}}$ is the representation of the Lie algebra $\widetilde{\overline{\boldsymbol{\Omega}}}$ with respect to the six translational velocity coordinates $\overline{\mathrm{w}}_i$.

3.3 Equations of motion of a free rigid body

In this section, we determine the governing equations of motion (EOM) in terms of $\overline{\mathbf{w}}$ and $\dot{\overline{\mathbf{w}}}$. In order to obtain a clear and simple structure of the underlying equations, we choose in the following the dimensions of the hexahedral as $\mathrm{l}_1 = \mathrm{l}_2 = \mathrm{l}_3 = 1$. To obtain the governing equations of motion we use the Gibbs-Appel equations [24, 25, 26]. Since we are searching for EOM depicted in a body-fixed frame, we have to calculate the partial derivatives of the Gibbs-function G with respect to the time derivative of the velocity coordinates $\dot{\overline{\mathbf{w}}}$ and equate the derivatives with the corresponding generalized forces $\bar{\mathbf{Q}}$

$$\frac{\partial G}{\partial \dot{\overline{\mathbf{w}}}} = \bar{\mathbf{Q}} \qquad \text{with} \qquad G = \frac{1}{2}\int_V \rho \bar{\dot{\mathbf{v}}}(\bar{\mathbf{y}}_p)^{\mathrm{T}} \bar{\dot{\mathbf{v}}}(\bar{\mathbf{y}}_p) dV \tag{44}$$

which need to be defined in the body-fixed frame too. Using Eq. (41), one can find that the Gibbs-function G can be written as a sum of three integrals

$$G = \frac{1}{2}\int_V \rho \bar{\dot{\mathbf{v}}}(\bar{\mathbf{y}}_p)^{\mathrm{T}} \bar{\dot{\mathbf{v}}}(\bar{\mathbf{y}}_p) dV = \frac{1}{2}\int_V \rho \dot{\overline{\mathbf{w}}}^{\mathrm{T}} \overline{\mathbf{N}}^{\mathrm{T}} \overline{\mathbf{N}} \dot{\overline{\mathbf{w}}} dV + \int_V \rho \dot{\overline{\mathbf{w}}}^{\mathrm{T}} \overline{\mathbf{N}}^{\mathrm{T}} \overline{\mathbf{P}} \overline{\mathbf{w}} dV + \frac{1}{2}\int_V \rho \overline{\mathbf{w}}^{\mathrm{T}} \overline{\mathbf{P}}^{\mathrm{T}} \overline{\mathbf{P}} \overline{\mathbf{w}} dV = G_1 + G_2 + G_3. \tag{45}$$

Since the vector $\dot{\overline{\mathbf{w}}}$ does not appear in the term G_3, the partial derivative of G_3 with respect to $\dot{\overline{\mathbf{w}}}$ vanishes. Therefore, we only make use of the first two terms G_1 and G_2, since their partial derivative with respect to $\dot{\overline{\mathbf{w}}}$ is obviously non zero. Therefore, G_1 and G_2 follows to

$$G_1 := \frac{1}{2} \dot{\overline{\mathbf{w}}}^{\mathrm{T}} \left(\int_V \rho \overline{\mathbf{N}}^{\mathrm{T}}(\bar{\mathbf{y}}_p) \overline{\mathbf{N}}(\bar{\mathbf{y}}_p) dV \right) \dot{\overline{\mathbf{w}}} = \frac{1}{2} \dot{\overline{\mathbf{w}}}^{\mathrm{T}} \overline{\mathbf{M}} \dot{\overline{\mathbf{w}}}, \tag{46}$$

$$G_2 := \dot{\overline{\mathbf{w}}}^{\mathrm{T}} \left(\int_V \rho \overline{\mathbf{N}}^{\mathrm{T}}(\bar{\mathbf{y}}_p) \overline{\mathbf{P}}(\bar{\mathbf{y}}_p, \overline{\mathbf{w}}) dV \right) \overline{\mathbf{w}} = \dot{\overline{\mathbf{w}}}^{\mathrm{T}} \overline{\mathbf{\Gamma}} \overline{\mathbf{w}}. \tag{47}$$

A further evaluation of the integrals in G_1 and G_2 lead us to the definition of a mass matrix $\overline{\mathbf{M}}$ and a velocity dependent matrix $\overline{\mathbf{\Gamma}}$

$$\overline{\mathbf{M}} := \int_V \rho \overline{\mathbf{N}}^{\mathrm{T}}(\bar{\mathbf{y}}_p) \overline{\mathbf{N}}(\bar{\mathbf{y}}_p) dV, \tag{48}$$

$$\overline{\mathbf{\Gamma}} := \int_V \rho \overline{\mathbf{N}}^{\mathrm{T}}(\bar{\mathbf{y}}_p) \overline{\mathbf{P}}(\bar{\mathbf{y}}_p, \overline{\mathbf{w}}) dV. \tag{49}$$

Since the interpolation matrix $\overline{\mathbf{N}}(\bar{\mathbf{y}}_p)$ depends only on the chosen reference point $\bar{\mathbf{y}}_p$, which is in a body-fixed frame constant, it follows that the mass matrix $\overline{\mathbf{M}}$ is also constant with respect to the body-fixed frame. The partial derivatives of G_1 and G_2 follow to

$$\frac{\partial G_1}{\partial \dot{\overline{\mathbf{w}}}} = \overline{\mathbf{M}} \dot{\overline{\mathbf{w}}} \quad \text{and} \quad \frac{\partial G_2}{\partial \dot{\overline{\mathbf{w}}}} = \overline{\mathbf{\Gamma}} \overline{\mathbf{w}}. \tag{50}$$

Adding up the partial derivatives of G_1 and G_2, the equations of motion follow to

$$\overline{\mathbf{M}} \dot{\overline{\mathbf{w}}} + \overline{\mathbf{\Gamma}} \overline{\mathbf{w}} = \overline{\mathbf{Q}}. \tag{51}$$

Not, that Eqs. (51) has been obtained without the use of rotation matrices or the angular velocity vector.

3.4 Generalized Forces

In order to obtain a complete formulation of the governing equations of motion (51), we define the corresponding generalized forces $\overline{\mathbf{Q}} \in \mathbb{R}^6$ here, using the principle of virtual power [27]. Since the obtained equations of motion are stated in a body-fixed frame, we need to derive the driving generalized forces with respect to a body-fixed frame. This can easily be done, since the principle of virtual power is invariant under a coordinate transformation. Therefore, the principle of virtual power follows to

$$\overline{\mathbf{Q}}^{\mathrm{T}}\delta\overline{\mathbf{w}} = \bar{\mathbf{f}}^{\mathrm{T}}\delta\overline{\mathbf{v}}(\bar{\mathbf{y}}_p) = \bar{\mathbf{f}}^{\mathrm{T}}\left(\frac{\partial\overline{\mathbf{v}}(\bar{\mathbf{y}}_p)}{\partial\overline{\mathbf{w}}}\delta\overline{\mathbf{w}}\right) = \bar{\mathbf{f}}^{\mathrm{T}}\left(\frac{\partial}{\partial\overline{\mathbf{w}}}\overline{\mathbf{N}}(\bar{\mathbf{y}}_p)\overline{\mathbf{w}}\delta\overline{\mathbf{w}}\right) = \bar{\mathbf{f}}^{\mathrm{T}}\overline{\mathbf{N}}(\bar{\mathbf{y}}_p)\delta\overline{\mathbf{w}}. \tag{52}$$

The generalized force vector $\overline{\mathbf{Q}}$, which acts on the point P (local position is given by vector $\bar{\mathbf{y}}_p$) of a rigid body as a result of the external body-fixed force vector $\bar{\mathbf{f}} \in \mathbb{R}^3$ therefore reads

$$\overline{\mathbf{Q}} = \overline{\mathbf{N}}^{\mathrm{T}}(\bar{\mathbf{y}}_p)\bar{\mathbf{f}}. \tag{53}$$

4 Relation to rotation dependent rigid body motion

In this section we show how the EOM (51) can be converted to the Newton-Euler EOM expressed with respect to a body-fixed frame. Furthermore, we present the results of the integrals of the mass matrix (48) and the integrals of the velocity depended matrix (49).

4.1 Relation between $\overline{\mathbf{w}}$ and $\overline{\mathbf{v}}_{\mathrm{L}}$

The vector $\overline{\mathbf{w}}$ can also be determined directly from the multiplication of the vector form of $\overline{\mathbf{v}}_{\mathrm{L}}$, see Eq. (10), with a matrix $\overline{\mathbf{D}} \in \mathbb{R}^{6\times 6}$

$$\overline{\mathbf{w}} = \begin{bmatrix} \overline{\mathrm{w}}_1 \\ \overline{\mathrm{w}}_2 \\ \overline{\mathrm{w}}_3 \\ \overline{\mathrm{w}}_4 \\ \overline{\mathrm{w}}_5 \\ \overline{\mathrm{w}}_6 \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{b}}_1^{\mathrm{T}} & \left(\tilde{\bar{\mathbf{y}}}_1\bar{\mathbf{b}}_1\right)^{\mathrm{T}} \\ \bar{\mathbf{b}}_2^{\mathrm{T}} & \left(\tilde{\bar{\mathbf{y}}}_2\bar{\mathbf{b}}_2\right)^{\mathrm{T}} \\ \bar{\mathbf{b}}_3^{\mathrm{T}} & \left(\tilde{\bar{\mathbf{y}}}_3\bar{\mathbf{b}}_3\right)^{\mathrm{T}} \\ \bar{\mathbf{b}}_4^{\mathrm{T}} & \left(\tilde{\bar{\mathbf{y}}}_4\bar{\mathbf{b}}_4\right)^{\mathrm{T}} \\ \bar{\mathbf{b}}_5^{\mathrm{T}} & \left(\tilde{\bar{\mathbf{y}}}_5\bar{\mathbf{b}}_5\right)^{\mathrm{T}} \\ \bar{\mathbf{b}}_6^{\mathrm{T}} & \left(\tilde{\bar{\mathbf{y}}}_6\bar{\mathbf{b}}_6\right)^{\mathrm{T}} \end{bmatrix} \overline{\mathbf{v}}_{\mathrm{L}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{\mathrm{l}_2}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{\mathrm{l}_2}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{\mathrm{l}_3}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & 0 & \frac{\mathrm{l}_3}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{\mathrm{l}_1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & \frac{\mathrm{l}_1}{\sqrt{2}} & 0 \end{bmatrix} \overline{\mathbf{v}}_{\mathrm{L}} = \overline{\mathbf{D}}\,\overline{\mathbf{v}}_{\mathrm{L}} \tag{54}$$

which is built from the direction and position vectors in Eq. (19-24). From Eq. (54) it follows that the vector form of the Lie algebra $\overline{\mathbf{v}}_{\mathrm{L}}$ can only be clearly reconstructed from the vector $\overline{\mathbf{w}}$ using the relation $\overline{\mathbf{v}}_{\mathrm{L}} = \overline{\mathbf{D}}^{-1}\overline{\mathbf{w}}$ if the matrix $\overline{\mathbf{D}}$ is invertible. In order to enable a bijective mapping between the rotation-free description of the rigid body motion presented here based on the six unified translational velocity coordinates $\overline{\mathrm{w}}$ and the description of the rigid body motion based on the translational and rotational velocities $\overline{\mathbf{D}}$ must be a regular matrix. The direction and position vectors selected in Eq. (19-24) fulfill this condition. Furthermore, if $\mathrm{l}_1 = \mathrm{l}_2 = \mathrm{l}_3 = 1$, the matrix $\overline{\mathbf{D}}$ formed with the direction and position vectors in Eq. (19-24) is orthogonal and its determinant is $\det(\overline{\mathbf{D}}) = -1$. The inverse matrix of $\overline{\mathbf{D}}$ follows to be $\overline{\mathbf{D}}^{-1} = \overline{\mathbf{D}}^{\mathrm{T}}$ and therefore

$$\overline{\mathbf{v}}_{\mathrm{L}} = \overline{\mathbf{D}}^{\mathrm{T}}\overline{\mathbf{w}}. \tag{55}$$

4.2 Mass matrix

If we choose the body's center of mass as reference point, one can observe that the integrals G_1 and G_2 in Eq. (45) can be simplified. The direct calculation of the integrals in G_1 and G_2 shows, that some matrices stand out within the general structure of integrals. Therefore, the mass matrix $\overline{\mathbf{M}}$ and the velocity dependent matrix $\overline{\mathbf{\Gamma}}$ can be written as a matrix product

$$\overline{\mathbf{M}} = \overline{\mathbf{D}}\begin{bmatrix} \bar{\mathbf{J}}_1 & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \bar{\mathbf{J}}_2 \end{bmatrix}\overline{\mathbf{D}}^{\mathrm{T}} \qquad \text{and} \qquad \overline{\mathbf{\Gamma}} = \overline{\mathbf{D}}\begin{bmatrix} \dot{\bar{\mathbf{E}}}\bar{\mathbf{J}}_1 & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \dot{\bar{\mathbf{E}}}\bar{\mathbf{J}}_2 \end{bmatrix}\overline{\mathbf{D}}^{\mathrm{T}} \tag{56}$$

where the diagonal elements of the 3×3-matrix $\overline{\mathbf{J}}_1$

$$\overline{\mathbf{J}}_1 = \int_V \rho \mathbf{I}_3 dV = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \tag{57}$$

contain the rigid body's mass m and the 3×3-matrix $\overline{\mathbf{J}}_2$

$$\overline{\mathbf{J}}_2(\bar{\mathbf{y}}_p) = \int_V \rho \tilde{\bar{\mathbf{y}}}_p \tilde{\bar{\mathbf{y}}}_p^T dV = \begin{bmatrix} \bar{\mathrm{J}}_{11} & \bar{\mathrm{J}}_{12} & \bar{\mathrm{J}}_{13} \\ \bar{\mathrm{J}}_{21} & \bar{\mathrm{J}}_{22} & \bar{\mathrm{J}}_{23} \\ \bar{\mathrm{J}}_{31} & \bar{\mathrm{J}}_{32} & \bar{\mathrm{J}}_{33} \end{bmatrix}. \tag{58}$$

represents the body's inertia tensor. If we use a diagonal inertia tensor

$$\overline{\mathbf{J}}_2(\bar{\mathbf{y}}_p) = \begin{bmatrix} \bar{\mathrm{J}}_{11} & 0 & 0 \\ 0 & \bar{\mathrm{J}}_{22} & 0 \\ 0 & 0 & \bar{\mathrm{J}}_{33} \end{bmatrix}, \tag{59}$$

the mass matrix $\overline{\mathbf{M}}$ exhibits a simple block diagonal structure

$$\overline{\mathbf{M}} = \frac{1}{2} \begin{bmatrix} \mathrm{m}+\bar{\mathrm{J}}_{33} & \bar{\mathrm{J}}_{33}-\mathrm{m} & 0 & 0 & 0 & 0 \\ \bar{\mathrm{J}}_{33}-\mathrm{m} & \mathrm{m}+\bar{\mathrm{J}}_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{m}+\bar{\mathrm{J}}_{11} & \bar{\mathrm{J}}_{11}-\mathrm{m} & 0 & 0 \\ 0 & 0 & \bar{\mathrm{J}}_{11}-\mathrm{m} & \mathrm{m}+\bar{\mathrm{J}}_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{m}+\bar{\mathrm{J}}_{22} & \bar{\mathrm{J}}_{22}-\mathrm{m} \\ 0 & 0 & 0 & 0 & \bar{\mathrm{J}}_{22}-\mathrm{m} & \mathrm{m}+\bar{\mathrm{J}}_{22} \end{bmatrix}. \tag{60}$$

Using a diagonal inertia tensor, the vector $\overline{\mathbf{\Gamma}}\,\overline{\mathbf{w}}$ containing the quadratic velocity vector reads

$$\overline{\mathbf{\Gamma}}\overline{\mathbf{w}} = \frac{1}{2\sqrt{2}} \begin{bmatrix} m(\overline{\mathrm{w}}_1+\overline{\mathrm{w}}_2)(\overline{\mathrm{w}}_4-\overline{\mathrm{w}}_3) + \left(\left(-\bar{\mathrm{J}}_{11}-\bar{\mathrm{J}}_{22}\right)(\overline{\mathrm{w}}_3+\overline{\mathrm{w}}_4) + m(\overline{\mathrm{w}}_5-\overline{\mathrm{w}}_6)\right)(\overline{\mathrm{w}}_5+\overline{\mathrm{w}}_6) \\ m(\overline{\mathrm{w}}_1+\overline{\mathrm{w}}_2)(\overline{\mathrm{w}}_3-\overline{\mathrm{w}}_4) - \left(\left(\bar{\mathrm{J}}_{11}-\bar{\mathrm{J}}_{22}\right)(\overline{\mathrm{w}}_3+\overline{\mathrm{w}}_4) + m(\overline{\mathrm{w}}_5-\overline{\mathrm{w}}_6)\right)(\overline{\mathrm{w}}_5+\overline{\mathrm{w}}_6) \\ m\left(\overline{\mathrm{w}}_1^2-\overline{\mathrm{w}}_2^2-(\overline{\mathrm{w}}_3+\overline{\mathrm{w}}_4)(\overline{\mathrm{w}}_5-\overline{\mathrm{w}}_6)\right) - \left(\bar{\mathrm{J}}_{22}-\bar{\mathrm{J}}_{33}\right)(\overline{\mathrm{w}}_1+\overline{\mathrm{w}}_2)(\overline{\mathrm{w}}_5+\overline{\mathrm{w}}_6) \\ -m\left(\overline{\mathrm{w}}_1^2-\overline{\mathrm{w}}_2^2-(\overline{\mathrm{w}}_3+\overline{\mathrm{w}}_4)(\overline{\mathrm{w}}_5-\overline{\mathrm{w}}_6)\right) - \left(\bar{\mathrm{J}}_{22}-\bar{\mathrm{J}}_{33}\right)(\overline{\mathrm{w}}_1+\overline{\mathrm{w}}_2)(\overline{\mathrm{w}}_5+\overline{\mathrm{w}}_6) \\ m\left(\overline{\mathrm{w}}_3^2-\overline{\mathrm{w}}_4^2-(\overline{\mathrm{w}}_1-\overline{\mathrm{w}}_2)(\overline{\mathrm{w}}_5+\overline{\mathrm{w}}_6)\right) + \left(\bar{\mathrm{J}}_{11}-\bar{\mathrm{J}}_{33}\right)(\overline{\mathrm{w}}_1+\overline{\mathrm{w}}_2)(\overline{\mathrm{w}}_3+\overline{\mathrm{w}}_4) \\ m\left(\overline{\mathrm{w}}_4^2-\overline{\mathrm{w}}_3^2+(\overline{\mathrm{w}}_1-\overline{\mathrm{w}}_2)(\overline{\mathrm{w}}_5+\overline{\mathrm{w}}_6)\right) + \left(\bar{\mathrm{J}}_{11}-\bar{\mathrm{J}}_{33}\right)(\overline{\mathrm{w}}_1+\overline{\mathrm{w}}_2)(\overline{\mathrm{w}}_3+\overline{\mathrm{w}}_4) \end{bmatrix}. \tag{61}$$

4.3 Relation to the Newton-Euler EOM

Since the mass matrix $\overline{\mathbf{M}}$ and the velocity dependent matrix $\overline{\mathbf{\Gamma}}$ can be written as matrix product as depicted in Eq. (56), the EOM (51) can be rewritten using matrix $\overline{\mathbf{D}}$,

$$\overline{\mathbf{M}}\dot{\overline{\mathbf{w}}} + \overline{\mathbf{\Gamma}}\overline{\mathbf{w}} = \mathbf{0}_{6\times1} \quad \Rightarrow \quad \overline{\mathbf{D}}\mathbf{M}\overline{\mathbf{D}}^{\mathrm{T}}\dot{\overline{\mathbf{w}}} + \overline{\mathbf{D}}\mathbf{\Psi}\mathbf{M}\overline{\mathbf{D}}^{\mathrm{T}}\overline{\mathbf{w}} = \mathbf{0}_{6\times1} \tag{62}$$

From Eq. (56) it follows, that the matrices $\mathbf{M}$ represent the usual mass matrix of the Newton-Euler EOM [6]. The matrix $\mathbf{\Psi}$ contains the skew symmetric representations of the local angular velocity vector,

$$\mathbf{M} = \begin{bmatrix} \overline{\mathbf{J}}_1 & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \overline{\mathbf{J}}_2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{\Psi} = \begin{bmatrix} \tilde{\overline{\mathbf{\Omega}}} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \tilde{\overline{\mathbf{\Omega}}} \end{bmatrix}. \tag{63}$$

Multiplication from left hand side with $\overline{\mathbf{D}}^{\mathrm{T}}$, gives

$$\mathbf{M}\overline{\mathbf{D}}^{\mathrm{T}}\dot{\overline{\mathbf{w}}} + \mathbf{\Psi}\mathbf{M}\overline{\mathbf{D}}^{\mathrm{T}}\overline{\mathbf{w}} = \mathbf{0}_{6\times1}. \tag{64}$$

Since $\dot{\overline{\mathbf{w}}} = \overline{\mathbf{D}}\dot{\overline{\mathbf{v}}}_L$ and $\overline{\mathbf{w}} = \overline{\mathbf{D}}\dot{\overline{\mathbf{v}}}_{\mathrm{L}}$ the equations of motion can be written as

$$\mathbf{M}\dot{\overline{\mathbf{v}}}_{\mathrm{L}} + \boldsymbol{\Psi}\mathbf{M}\overline{\mathbf{v}}_{\mathrm{L}} = \mathbf{0}_{6\times 1}. \tag{65}$$

Splitting up the EOM (65) into a term for the translational motion and the rotational motion leads to the well known Newton-Euler equations, e.g. see [12, 13]

$$\overline{\mathbf{J}}_1\dot{\overline{\mathbf{U}}} + \widetilde{\overline{\boldsymbol{\Omega}}}\overline{\mathbf{J}}_1\overline{\mathbf{U}} = \mathbf{0}_{3\times 1} \tag{66}$$

$$\overline{\mathbf{J}}_2\dot{\overline{\boldsymbol{\Omega}}} + \widetilde{\overline{\boldsymbol{\Omega}}}\overline{\mathbf{J}}_2\overline{\boldsymbol{\Omega}} = \mathbf{0}_{3\times 1}. \tag{67}$$

5 Time integration of the unified local velocity coordinates

In order to obtain the current position and orientation of the rigid body with respect to the reference configuration from the unified velocity coordinates $\overline{\mathrm{w}}$, we need to find a proper representation of the Lie algebra $\widetilde{\overline{\mathbf{v}}}_{\mathrm{L}}$ or its vector representation $\overline{\mathbf{v}}_{\mathrm{L}}$ and furthermore of the inverse tangent operator $\mathbf{T}_{\mathrm{SE}(3)}^{-1}$ with respect to $\overline{\mathrm{w}}$. Once this is achieved, we only need to apply a proper time integration method to the incremental motion vector differential Eq. (11) and apply its solution to Eq. (12) to obtain the current position and orientation of the rigid body. Therefore, the goal of this section is, to find a proper representation of Lie algebra $\widetilde{\overline{\mathbf{v}}}_{\mathrm{L}}$ as well as a proper representation of the inverse tangent operator $\mathbf{T}_{\mathrm{SE}(3)}^{-1}$ and furthermore to describe the numerical time integration of the governing equations of motion (51), the incremental motion vector ODE (11) and the kinematic differential Eq. (6).

5.1 Lie algebra corresponding to $\overline{\mathrm{w}}$

Since the Lie algebra $\widetilde{\overline{\mathbf{v}}}_{\mathrm{L}}$ in the kinematic differential Eq. (6) is a function of the translational velocity vector $\overline{\mathbf{U}}$ and the angular velocity vector $\overline{\boldsymbol{\Omega}}$, we need to find a corresponding representation of the Lie algebra with respect to the velocity coordinates $\overline{\mathrm{w}}$ in order to be able to apply the solution approach depicted in Eq. (12). This corresponding representation of the Lie algebra can easily be found, since the translational velocity of the reference point can be determined with respect to $\overline{\mathbf{w}}$ using Eq. (26)

$$\overline{\mathbf{U}} = \overline{\mathbf{N}}(\mathbf{0}_{1\times 3})\overline{\mathbf{w}}. \tag{68}$$

Since, the local angular velocity tensor $\widetilde{\overline{\boldsymbol{\Omega}}}$ can be expressed with matrix $\overline{\mathbf{E}}$, the representation to the Lie algebra $\mathfrak{so}(3)$ with respect to $\overline{\mathbf{w}}$ corresponds in case that the basis vectors represent the standard orthogonal basis to

$$\widetilde{\overline{\boldsymbol{\Omega}}} = \overline{\mathbf{E}}(\overline{\mathbf{w}}). \tag{69}$$

Therefore, the Lie algebra $\widetilde{\overline{\mathbf{v}}}_{\mathrm{L}}(\overline{\mathbf{w}})$ corresponding to $\mathbf{H} \in \mathrm{SE}(3)$ is given by the matrix

$$\widetilde{\overline{\mathbf{v}}}_{\mathrm{L}}(\overline{\mathbf{w}}) := \begin{bmatrix} \overline{\mathbf{E}}(\overline{\mathbf{w}}) & \overline{\mathbf{N}}(\mathbf{0}_{1\times 3})\overline{\mathbf{w}} \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-\overline{\mathrm{w}}_1-\overline{\mathrm{w}}_2}{\sqrt{2}} & \frac{\overline{\mathrm{w}}_5+\overline{\mathrm{w}}_6}{\sqrt{2}} & \frac{\overline{\mathrm{w}}_1-\overline{\mathrm{w}}_2}{\sqrt{2}} \\ \frac{\overline{\mathrm{w}}_1+\overline{\mathrm{w}}_2}{\sqrt{2}} & 0 & \frac{-\overline{\mathrm{w}}_3-\overline{\mathrm{w}}_4}{\sqrt{2}} & \frac{\overline{\mathrm{w}}_3-\overline{\mathrm{w}}_4}{\sqrt{2}} \\ \frac{-\overline{\mathrm{w}}_5-\overline{\mathrm{w}}_6}{\sqrt{2}} & \frac{\overline{\mathrm{w}}_3+\overline{\mathrm{w}}_4}{\sqrt{2}} & 0 & \frac{\overline{\mathrm{w}}_5-\overline{\mathrm{w}}_6}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{70}$$

Since the lie algebra $\mathfrak{g}$ is isomorphic to $\mathbb{R}^n$ by the invertible linear map (9), the vector form $\overline{\mathbf{v}}_{\mathrm{L}}$ of the lie algebra $\overline{\mathbf{v}}_{\mathrm{L}}$ can be formed from the matrix-vector multiplication

$$\overline{\mathbf{v}}_{\mathrm{L}} := \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \overline{\mathrm{w}}_1 \\ \overline{\mathrm{w}}_2 \\ \overline{\mathrm{w}}_3 \\ \overline{\mathrm{w}}_4 \\ \overline{\mathrm{w}}_5 \\ \overline{\mathrm{w}}_6 \end{bmatrix} = \overline{\mathbf{D}}^{\mathrm{T}}\overline{\mathbf{w}}. \tag{71}$$

5.2 Inverse tangent operator

The differential equation for the incremental motion vector $\dot{\mathbf{n}}$ is calculated as already shown in Eq. (11) from the matrix vector multiplication of the tangential operator by the vector form of the Lie algebra. The relationship between the vector form of the lie algebra $\overline{\mathbf{v}}_{\mathrm{L}}$ and the vector of the unified coordinates $\overline{\mathbf{w}}$ is established by the direction vectors $\bar{\mathbf{b}}$ and the position vectors $\bar{\mathbf{y}}_P$ in the form of the inverse transformation matrix $\overline{\mathbf{D}}^{\mathrm{T}}$ via the relationship $\overline{\mathbf{v}}_{\mathrm{L}} = \overline{\mathbf{D}}^{\mathrm{T}}\overline{\mathbf{w}}$. The tangential operator $\overline{\mathbf{T}}^{-1}_{\mathrm{SE(3)}}$ matching the unified coordinates can thus be determined by matrix multiplication

$$\overline{\mathbf{T}}^{-1}_{\mathrm{SE(3)}}(\mathbf{n}) = \mathbf{T}^{-1}_{\mathrm{SE(3)}}(\mathbf{n})\overline{\mathbf{D}}^{\mathrm{T}} \tag{72}$$

The tangential operator $\overline{\mathbf{T}}^{-1}_{\mathrm{SE(3)}}$ can be written similar to the tangential operator $\mathbf{T}^{-1}_{\mathrm{SE(3)}}$ in closed [22] form as

$$\overline{\mathbf{T}}^{-1}_{\mathrm{SE(3)}}(\mathbf{n}) = \overline{\mathbf{D}}^{\mathrm{T}} + \frac{1}{2}\mathbf{T}_1(\mathbf{n}) + \mathbf{K}(\mathbf{n})\mathbf{T}_2(\mathbf{n}) \tag{73}$$

with

$$\mathbf{T}_1(\mathbf{n}) = \begin{bmatrix} \tilde{\mathbf{n}}_{46} & \tilde{\mathbf{n}}_{13} \\ \mathbf{0}_{3\times 3} & \tilde{\mathbf{n}}_{46} \end{bmatrix} \overline{\mathbf{D}}^{\mathrm{T}}, \tag{74}$$

where $\mathbf{n} = \begin{bmatrix} \mathrm{n}_1 & \mathrm{n}_2 & \mathrm{n}_3 & \mathrm{n}_4 & \mathrm{n}_5 & \mathrm{n}_6 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \mathbf{n}_{13}^{\mathrm{T}} & \mathbf{n}_{46}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$. The coefficient matrix $\mathbf{K}(\mathbf{n})$ is defined via Eqs.

$$\mathbf{K}(\boldsymbol{n}) = \begin{bmatrix} \mathrm{k}_1(\mathbf{n})\mathbf{I}_3 & \mathrm{k}_2(\mathbf{n})\mathbf{I}_3 \\ \mathbf{0}_3 & \mathrm{k}_1(\mathbf{n})\mathbf{I}_3 \end{bmatrix}, \tag{75}$$

$$\mathrm{k}_1(\mathbf{n}) = \frac{1}{\|\mathbf{n}_{46}\|^2}\left(1 - \frac{\|\mathbf{n}_{46}\|}{2}\cot\left(\frac{\|\mathbf{n}_{46}\|}{2}\right)\right), \quad \text{and} \tag{76}$$

$$\mathrm{k}_2(\mathbf{n}) = \frac{\mathbf{n}_{46}^T\mathbf{n}_{13}}{2\|\mathbf{n}_{46}\|^4}\left(\frac{\|\mathbf{n}_{46}\|^2}{1-\cos(\|\mathbf{n}_{46}\|)} + \|\mathbf{n}_{46}\|\cot\left(\frac{\|\mathbf{n}_{46}\|}{2}\right) - 4\right). \tag{77}$$

The matrix $\mathbf{T}_2(\mathbf{n})$ is defined using the Lie bracket $\lceil , \rceil$ see [15],

$$\mathbf{T}_2(\mathbf{n}) = \begin{bmatrix} \lceil \tilde{\mathbf{n}}_{13}, \tilde{\mathbf{n}}_{46} \rceil & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \tilde{\mathbf{n}}_{46}^2 \end{bmatrix} \overline{\mathbf{D}}^{\mathrm{T}}. \tag{78}$$

5.3 Integration scheme

We focus next on a numerical time integration scheme, which can be used to solve the differential equations

$$\dot{\overline{\mathbf{w}}}_{\mathrm{i+1}} = \overline{\mathbf{M}}^{-1}\left(\overline{\mathbf{Q}}_{\mathrm{i}} - \overline{\boldsymbol{\Gamma}}_{\mathrm{i}}\overline{\mathbf{w}}_{\mathrm{i}}\right) = \mathrm{f}(\mathbf{H}_0, \overline{\mathbf{w}}_0, \mathrm{t}_0), \tag{79}$$

$$\dot{\mathbf{n}}_{\mathrm{i+1}} = \overline{\mathbf{T}}^{-1}_{\mathrm{SE(3)}}(\mathbf{n}_{\mathrm{i}})\overline{\mathbf{w}}_{\mathrm{i+1}}. \tag{80}$$

Once a solution of Eq. (80) is obtained for a specific time step (i $\rightarrow$ i+1), the corresponding SE(3) group element update is obtained from the update formula

$$\mathbf{H}_{\mathrm{i+1}} = \mathbf{H}_{\mathrm{i}}\exp_{\mathrm{SE(3)}}(\widetilde{\mathbf{n}}_{\mathrm{i+1}}). \tag{81}$$

For the purpose of numerical time integration, we adopt a recently presented 4th-order Runge-Kutta time integration scheme proposed by Terze et al. [20], such that it can be applied to our formulation. Starting with values at the previous step, $\overline{\mathbf{w}}_{\mathrm{i}}, \Delta\mathrm{t}_{\mathrm{i}}, \mathbf{H}_{\mathrm{i}}, \mathbf{n}_{\mathrm{i}}$, the slope estimations are obtained by

$$\mathbf{K}_1 = \Delta\mathrm{t}\,\overline{\mathbf{T}}^{-1}_{\mathrm{SE(3)}}(\mathbf{n}_{\mathrm{i}})\,\overline{\mathbf{w}}_{\mathrm{i}}, \qquad \mathbf{k}_1 = \Delta\mathrm{t}\,\mathrm{f}(\mathbf{H}_{\mathrm{i}}, \overline{\mathbf{w}}_{\mathrm{i}}, \mathrm{t}_{\mathrm{i}}), \tag{82}$$

$$\mathbf{K}_2 = \Delta\mathrm{t}\,\overline{\mathbf{T}}^{-1}_{\mathrm{SE(3)}}\left(\frac{1}{2}\mathbf{K}_1\right)\left(\overline{\mathbf{w}}_{\mathrm{i}} + \frac{1}{2}\mathbf{k}_1\right), \qquad \mathbf{k}_2 = \Delta\mathrm{t}\,\mathrm{f}\left(\mathbf{H}_{\mathrm{i}}, \overline{\mathbf{w}}_{\mathrm{i}} + \frac{1}{2}\mathbf{k}_1, \mathrm{t}_{\mathrm{i}} + \frac{1}{2}\Delta\mathrm{t}\right), \tag{83}$$

$$\mathbf{K}_3 = \Delta\mathrm{t}\,\overline{\mathbf{T}}^{-1}_{\mathrm{SE(3)}}\left(\frac{1}{2}\mathbf{K}_2\right)\left(\overline{\mathbf{w}}_{\mathrm{i}} + \frac{1}{2}\mathbf{k}_2\right), \qquad \mathbf{k}_3 = \Delta\mathrm{t}\,\mathrm{f}\left(\mathbf{H}_{\mathrm{i}}, \overline{\mathbf{w}}_{\mathrm{i}} + \frac{1}{2}\mathbf{k}_2, \mathrm{t}_{\mathrm{i}} + \frac{1}{2}\Delta\mathrm{t}\right), \tag{84}$$

$$\mathbf{K}_4 = \Delta\mathrm{t}\,\overline{\mathbf{T}}^{-1}_{\mathrm{SE(3)}}(\mathbf{K}_3)(\overline{\mathbf{w}}_{\mathrm{i}} + \mathbf{k}_3), \qquad \mathbf{k}_4 = \Delta\mathrm{t}\,\mathrm{f}(\mathbf{H}_{\mathrm{i}}, \overline{\mathbf{w}}_{\mathrm{i}} + \mathbf{k}_3, \mathrm{t}_{\mathrm{i}} + \Delta\mathrm{t}). \tag{85}$$

The four coefficients $\mathbf{K}_1,\ldots,\mathbf{K}_4$ represent the slopes within one time step (i $\rightarrow$ i+1) in the solution process of the incremental motion vector ODE (80) and the four coefficients $\mathbf{k}_1,\ldots,\mathbf{k}_4$ contain the slopes within one time step in the solution process of the EOM (79). The quantity Δt represents the time step size in the integration process. The following two equations

$$\overline{\mathbf{w}}_{\mathrm{i+1}} = \overline{\mathbf{w}}_{\mathrm{i}} + \frac{1}{6}\left(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4\right), \tag{86}$$

$$\mathbf{H}_{\mathrm{i+1}} = \mathbf{H}_{\mathrm{i}} \exp_{\mathrm{SE(3)}}\left(\frac{1}{6}\left(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_1\right)\right), \tag{87}$$

update the quantities $\overline{\mathbf{w}}$ and $\mathbf{H}$. Note, that the incremental motion vector is set to $\mathbf{n}_0 = \mathbf{0}_{6\times1}$ for i = 0.

6 Numerical example

As an example of usage, we present the rotation of a rigid body about an axis close to its unstable axis of rotation. No external torques are applied. This example is taken from [20, 28]. The rigid body is a box with an inertia tensor with respect to the body's center of mass equal to $\overline{\mathbf{J}}_2 = \mathrm{diag}(5.2988, 1.1775, 4.3568)$. All quantities are expressed in standard units. A unstable rotation about the third axis is expected, since $\overline{\mathrm{J}}_{11} > \overline{\mathrm{J}}_{22} > \overline{\mathrm{J}}_{33}$. Since we focused the analysis of the six unified local velocity coordinates on the hexahedral projection with side lengths ($\mathrm{l}_1 = \mathrm{l}_2 = \mathrm{l}_3 = 1$), the hexahedral projection vectors $\bar{\mathbf{b}}_i$ and the projection points $\bar{\mathbf{y}}_i$, $i = 1...6$, of Eqs. (19-24), are used to obtain six unified local velocity coordinates from the body's velocity state,

$$\overline{\mathbf{v}}_0 = \begin{bmatrix}\overline{\mathbf{U}}_0^{\mathrm{T}} & \overline{\boldsymbol{\Omega}}_0^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}0 & 0 & 0 & 0.01 & 0 & 100\end{bmatrix}^{\mathrm{T}}, \tag{88}$$

where $\overline{\mathbf{U}}_0 = \mathbf{0}_{1\times3}$ represents the body's local center of mass velocity vector and $\overline{\boldsymbol{\Omega}}_0 = \begin{bmatrix}0.01 \ 0 \ 100\end{bmatrix}^{\mathrm{T}}$ the body's local angular velocity vector corresponding to the values used in [20, 28]. Therefore, the initial conditions are set to

$$\overline{\mathbf{w}}_0 = \begin{bmatrix}\frac{100}{\sqrt{2}} & \frac{100}{\sqrt{2}} & \frac{0.01}{\sqrt{2}} & \frac{0.01}{\sqrt{2}} & 0 & 0\end{bmatrix}^{\mathrm{T}} \tag{89}$$

and $\mathbf{R}_0 = \mathbf{I}_3$ and $\mathbf{x}_0 = \mathbf{0}_{3\times1}$. The rotation matrix $\mathbf{R}_0$ represents the body's initial orientation and the vector $\mathbf{x}_0$ the body's initial position with respect to the reference frame. Therefore, the initial conditions in terms of an element of SE(3) follow to

$$\mathbf{H}_0 = \begin{bmatrix}\mathbf{R}_0 & \mathbf{x}_0 \\ \mathbf{0} & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}. \tag{90}$$

The initial conditions $\overline{\mathbf{w}}_0$ represent $\overline{\mathbf{U}}_0$ and $\overline{\boldsymbol{\Omega}}_0$, since by definition, the velocity state of a rigid body is expressed in case of a hexahedral projection by $\overline{\mathbf{w}}_0 = \overline{\mathbf{D}}\,\overline{\mathbf{v}}_0$. Since no external forces or torques are applied, the equations of motion follow to

$$\overline{\mathbf{M}}\dot{\overline{\mathbf{w}}} + \overline{\boldsymbol{\Gamma}}\overline{\mathbf{w}} = \mathbf{0}_{6\times1} \tag{91}$$

Despite that no translational motion is studied in this example, we set the body's mass $m = 1$ in Eq. (60) for the sake of completeness.

For a better understanding of the simulation results, the position of a point P, which position vector is given in the body-fixed frame by $\bar{\mathbf{p}} = \begin{bmatrix}0 \ 0 \ 1\end{bmatrix}^{\mathrm{T}}$, is tracked during rotation. Numerical simulations are conducted in the time domain of one second. In order to obtain a reference solution, we use the time integration scheme without explicitly imposing the unit-length constraint proposed by Terze et al. [20], which we will abbreviate in the following figures with $f1$. Our unified local velocity coordinate formulation will be abbreviated with *ulvc* in the following figures. Since all methods converged to the same solution (up to machine precision) for a time step of $h = 1e-5$, results obtained with that time step can be considered as a reference. The trajectory of $\bar{\mathbf{p}}$, expressed

Fig. 7: Spatial trajectory of point $\bar{\mathbf{p}}$

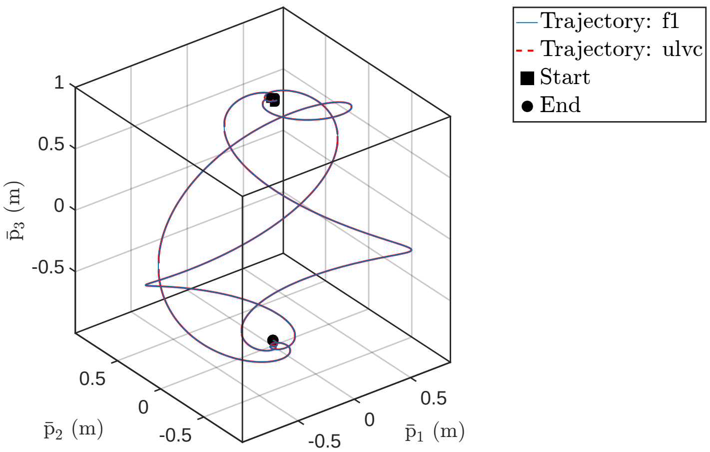
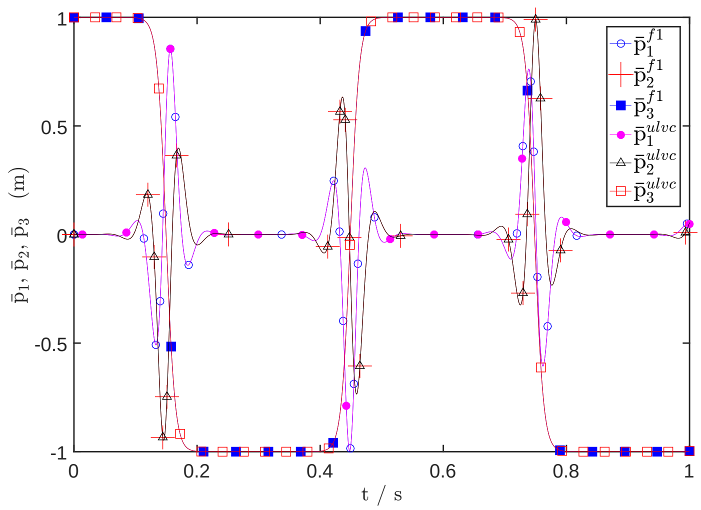


Fig. 8: Position of point $\bar{\mathbf{p}}$



in an inertial frame, is shown in the three-dimensional plot in Fig. 7, while its components as a function of time are shown in Fig. 8. The position of the point $\bar{\mathbf{p}}$ is changing between $[0\ 0\ 1]^{\mathrm{T}}$ and $[0\ 0\ -1]^{\mathrm{T}}$. Components of the body's angular velocity vector $\overline{\boldsymbol{\Omega}}$ as a function of time are shown in Fig. 9. As expected, the body's rotation is unstable after rotating about an axis close to the third axis, the body flips over and continues its rotation as one can observe in Fig. 9. The condition that the determinant of the rotation matrix $\det(\mathbf{R}) = +1$ is shown as a function of time in Fig. 10. Finally, Fig. 11 illustrates convergence based the norm of the error of the angular velocity error $\|\overline{\boldsymbol{\Omega}} - \overline{\boldsymbol{\Omega}}_{\text{converged}}\|_2$ and Fig. (12) illustrates the determinant of the rotation matrix $\det(\mathbf{R}) = +1$ for decreasing values of the integration step size $h(1/10,\ 1/40,\ 1/80,\ 1/160,\ 1/320,\ 1/640,\ 1/1280,\ 1/2560,\ 1/5120)$. The reference solution is obtained with the time step $h = 1e-5$. The convergence plots are displayed in logarithmic scale. As shown in Fig. 11 the unified local velocity coordinate formulation exhibits the same convergence and accuracy as the method proposed by Terze et al.[20] (largest difference in the norm of the angular velocity vector error is 0.0581), since both integration schemes are based on a fourth order explicit Runge-Kutta method. As depicted in Fig. 12, the determinant of the rotation matrix condition $\det(\mathbf{R}) - 1 = 0$ stays for both methods under 10^{-13}. The unified local velocity coordinate formulation shows a slightly better accuracy in the condition $\det(\mathbf{R}) - 1 = 0$ than the one obtained with the method proposed by [20], whereas the method presented in [20] shows a slightly higher convergence rate. The better accuracy of the unified local velocity coordinate formulation compared to the one obtained with the method of Terze et al. may be explained by the fact, that we use directly the exponential map on SO(3) to obtain a rotation matrix from which we compute the determinant, whereas in the method of Terze et al.

Fig. 9: Body angular velocity

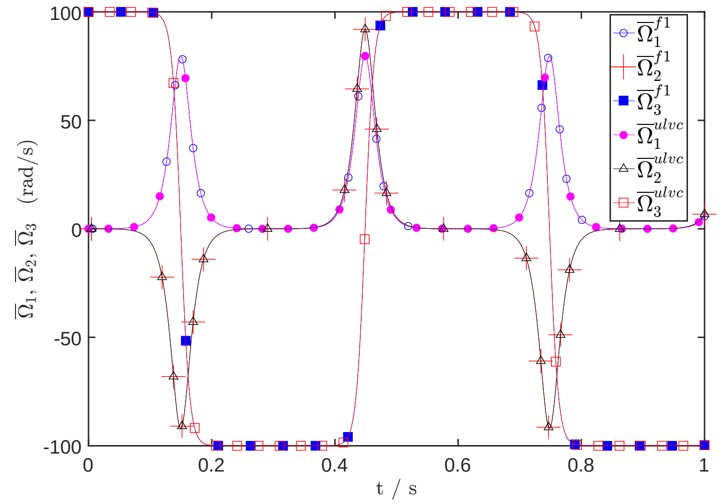


Fig. 10: Determinant of rotation matrix

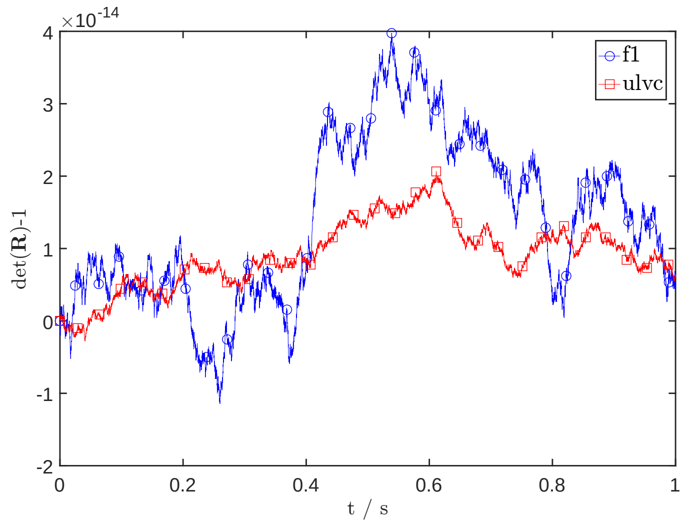
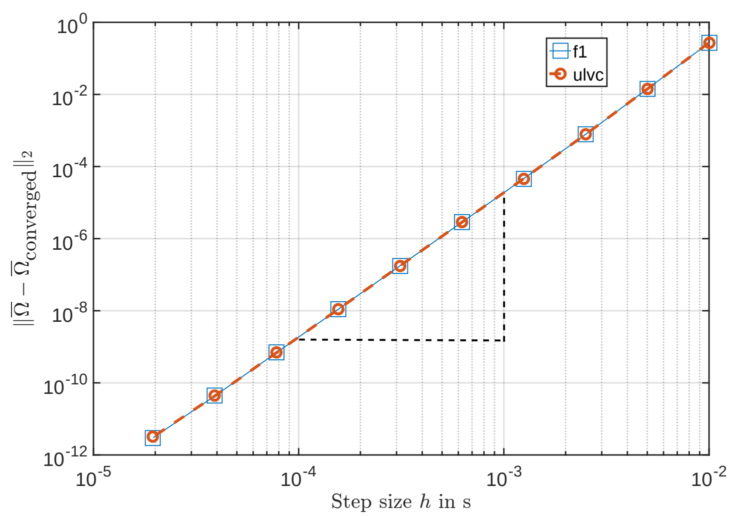
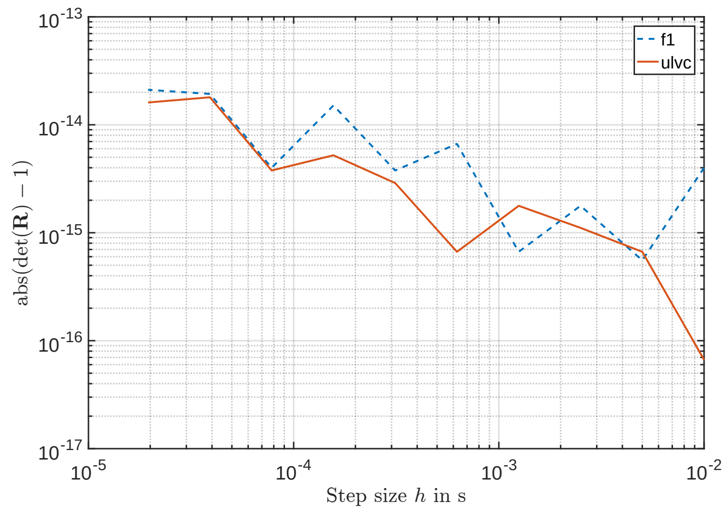


Fig. 11: Convergence in the norm of the angular velocity error. The triangle in the figure represents a gradient triangle. As can be seen from the figure, the norm of the error of the solution of the local angular velocity vector converges with fourth order.



one have to first compute a unit quaternion with which one can construct a rotation matrix and after that one can compute its determinant.

Fig. 12: Curve of the rotation matrix condition $\det(\mathbf{R}) = +1$ with respect to time integration step size



7 Conclusions

Concluding, we found a formulation to describe spatial rigid body motion in terms of non-redundant, homogeneous local velocity coordinates. We obtain equations of motion with a simple structure, which can be integrated using adopted Lie-group time integration schemes. Currently, the proposed approach offers no noticeable computational advantages compared to state of the art formulations but it promises to be useful in cases, in which one does not want to distinguish between translational and rotational motion.

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