The motions of the celt on a horizontal plane with viscous friction

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ABSTRACT — *The problem of the motion of a celt on a fixed horizontal plane with viscous friction is considered. On the plane of the parameters of the problem, regions of stability of uniform rotations about the vertical are constructed. The dynamics of transient processes from unstable motions to stable ones is studied.*

1 Introduction

The Celt is a convex solid body, one of its principal central axis of inertia being perpendicular to the surface of the body, and the directions of the principal curvatures of the surface at the point of intersection with this axis are not parallel to the other two principal axes. It is well known that the stability of the rotations of this body around the vertical axis depends from the direction of rotation. In most of the papers devoted to this property, the non-holonomic formulation of the problem is considered;it is assumed that the velocity of the point of contact between the body and the plane is zero (see, for example, [1], [2]). In the paper [3] the motion of the celt on a plane with friction is considered, and the consistency of this formulation of the problem with full-scale experiments is confirmed.

In the present paper, the investigation of [1] is continuing, in which it is assumed that the viscous friction force acts on the stone from the side of the plane. This model of friction allows us to carry out not only numerical, but also analytical studies in the problem. In addition, when the coefficient of viscous friction strives for infinity, the force of viscous friction is realized the non-holonomic constraint [4].

2 Statement of the problem

We will introduce the following variables: $\mathbf{v}$ is the velocity of the mass centre of the celt, $\boldsymbol{\omega}$ is its angular velocity and $\boldsymbol{\gamma}$ is the unit vector of the rising vertical. The slipping velocity is given by the relation $\mathbf{u} = \mathbf{v} + [\boldsymbol{\omega}, \mathbf{r}]$, where $\mathbf{r}$ is the radius vector of the point of contact of the body with the plane, defined by the equation $\boldsymbol{\gamma} = -\operatorname{grad} \mathrm{f}(\mathbf{r})/|\operatorname{grad} \mathrm{f}(\mathbf{r})|$.

A gravitational force $\mathbf{P} = -mg\boldsymbol{\gamma}$, the normal component of the reaction of the support plane $\mathbf{N} = N\boldsymbol{\gamma}$ and the friction force $\mathbf{F} = -mk\mathbf{u}$ (k is the coefficient of viscous friction) act on the celt. The equations of its motion in the moving coordinates have the form

$$m\dot{\mathbf{v}} + [\boldsymbol{\omega}, m\mathbf{v}] = (N - mg)\boldsymbol{\gamma} + \mathbf{F} \tag{1}$$

$$\mathbf{J}\dot{\boldsymbol{\omega}} + [\boldsymbol{\omega}, \mathbf{J}\boldsymbol{\omega}] = [\mathbf{r}, N\boldsymbol{\gamma} + \mathbf{F}] \tag{2}$$

$$\dot{\boldsymbol{\gamma}} + [\boldsymbol{\omega}, \boldsymbol{\gamma}] = 0 \tag{3}$$

$$(\mathbf{v} + [\boldsymbol{\omega}, \mathbf{r}], \boldsymbol{\gamma}) = 0 \tag{4}$$

where $\mathbf{J} = \operatorname{diag}(\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3)$ is the central inertia tensor or the celt. Eq. ((1)) is the theorem of the change of the momentum of the celt, Eq. (2) is the theorem of the change of the angular momentum, Eq. (3) is the condition for the vector $\boldsymbol{\gamma}$ to be constant in an absolute coordinates, and Eq. (4) is the condition for the celt to be in contact with

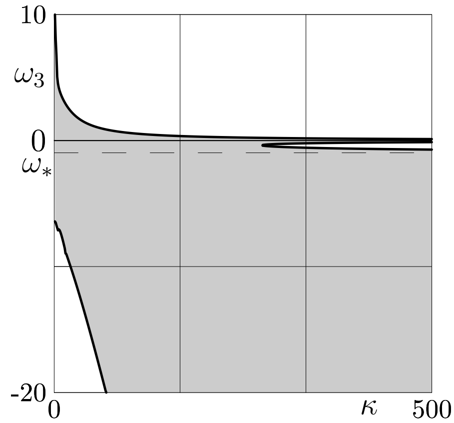

Fig. 1: $A_1 = 3\ kg \cdot m^2$, $A_2 = 4\ kg \cdot m^2$, $A_3 = 5\ kg \cdot m^2$, $a_1 = 5\ m$, $a_2 = 4\ m$, $a_3 = 3\ m$, $m = 1\ kg$, $\delta = 0.75$; $J = 7\ kg \cdot m^2$, $\omega_* = -0.97\ s^{-1}$

the supporting plane. The system (1)–(4) is closed with respect to the variables $\mathbf{v}$, $\boldsymbol{\omega}$, γ and N. From this system, the normal reaction of the supporting plane is determined

$$N = m\,(g + ([\mathbf{r},\dot{\boldsymbol{\omega}}] + [\dot{\mathbf{r}},\boldsymbol{\omega}],\gamma)) + ([\boldsymbol{\omega},r],[\boldsymbol{\omega},\gamma])) \tag{5}$$

then the system (1)–(3), taking into account Eq. (5) is considered

The resulting system of equations has solutions of the form

$$v_1 = v_2 = v_3 = 0, \quad \gamma_1 = \gamma_2 = 0, \quad \gamma_3 = 1, \quad \omega_1 = \omega_2 = 0, \quad \omega_3 = \omega\ (\omega \in R). \tag{6}$$

They correspond to uniform rotations of the elt around the principal axis of inertia which is normal to its surface and coincides with the vertical. The equation of the body surface at $\gamma_3 = 1$ can be represented in the form

$$f(\mathbf{r}) = x_3 + a_3 - \frac{(x_1\cos\delta + x_2\sin\delta)^2}{2a_1} - \frac{(x_1\sin\delta - x_2\cos\delta)^2}{2a_2} + O_3(x_1,x_2),$$

where $a_1,\ a_2$ – are the main radii of curvature of the body surface at the point of contact, a_3 – is the height of the center of mass, δ — is the angle between the vectors of principal curvatures and principal axes (for celt the relations are satisfied $A_1 \neq A_2,\ a_1 \neq a_2,\ \delta \neq 0\ (\mathrm{mod}\,\pi/2)$).

3 Stability conditions

Linearized equations of perturbed motion of the system in the neighborhood of solutions Eq. (6) and the corresponding characteristic equation are in [1].In the case of non-holonomic statement of the problem ($k \to +\infty$) the stability conditions have the form [1],[2]

$$A_1 < A_2 < A_3, \quad a_1 > a_2 > a_3, \quad 0 < \delta < \frac{\pi}{2} \tag{7}$$

$$J = (A_1 + A_2 - A_3)\left(\frac{a_1}{a_3} + \frac{a_2}{a_3} - 2\right) - ma_3^2\left(4 - 3\left(\frac{a_1}{a_3} + \frac{a_2}{a_3}\right) + 2\frac{a_1}{a_3}\frac{a_2}{a_3}\right) > 0 \tag{8}$$

$$\omega < 0,\ \omega^2 > \omega_*^2 = \frac{mg}{Ja_3}(a_1 - a_3)(a_2 - a_3) \tag{9}$$

Eq. (7) means that the rotation occurs around the axis of the greatest moment of inertia, and the corresponding equilibrium ($\omega = 0$) is stable. Eq. (8) imposes constraints on geometric and dynamic parameters of the body. Eq. (9) means that only rotations in the negative direction and with a enough large angular velocity are stable.

In the case of an arbitrary coefficient of viscous friction, the linearized equations of the perturbed motion of the system in the neighborhood of the solutions Eq. (6) are rather cumbersome [1] and the analytical analysis of

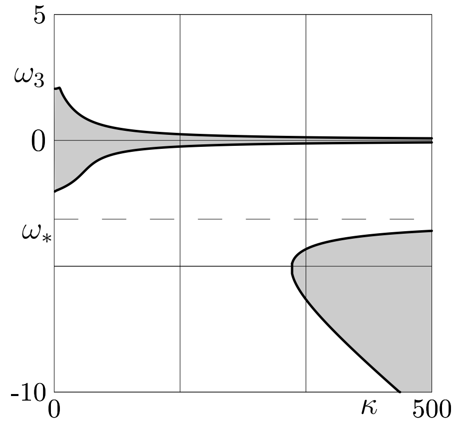

Fig. 2: $A_1 = 3\ kg \cdot m^2$, $A_2 = 4\ kg \cdot m^2$, $A_3 = 5\ kg \cdot m^2$, $a_1 = 5\ m$, $a_2 = 4\ m$, $a_3 = 2\ m$, $m = 1\ kg$, $\delta = 0.75$; $J = 3\ kg \cdot m^2$, $\omega_* = -3.13\ s^{-1}$

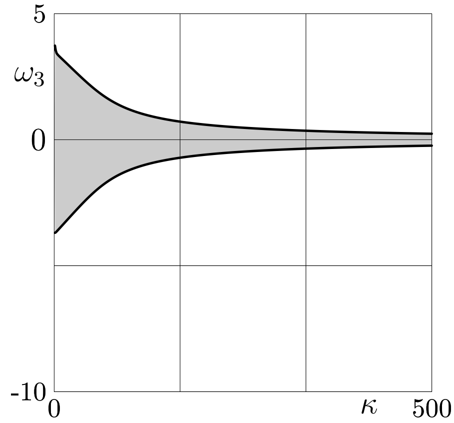

Fig. 3: $A_1 = 3\ kg \cdot m^2$, $A_2 = 4\ kg \cdot m^2$, $A_3 = 5\ kg \cdot m^2$, $a_1 = 5\ m$, $a_2 = 4\ m$, $a_3 = 2\ m$, $m = 3\ kg$, $\delta = 0.75$, $J = -1\ kg \cdot m^2$

stability conditions is difficult. At Fig. 1, Fig. 2, Fig. 3 the stability regions for some parameters of the problem are given. In the case shown at Fig. 1, for sufficiently large friction coefficients, there are two regions of stability. One corresponds to a non-holonomic statement of the problem (below the dotted line), the second is located in a neighborhood of zero. As k decreases, these regions merge into one, containing almost all negative values of ω and a small range of positive values ω. At very small k he stability region have some symmetry with respect to the horizontal.

In the case shown at Fig. 2 (which differs from the previous only by the height of the center of mass), the stability region is divided into two parts: the region corresponding to the non-holonomic case and the region in the neighborhood of equilibrium. In the case shown at Fig. 3 (body mass increases) only the region of stability in the neighborhood of equilibrium remains.

4 Numerical experiments

Numerical experiments were carried out for a celt with parameters

$$A_1 = 0.058 \cdot 10^{-3}\ kg \cdot m^2,\ A_2 = 0.44 \cdot 10^{-3}\ kg \cdot m^2,\ A_3 = 0.49 \cdot 10^{-3}\ kg \cdot m^2$$
$$a_1 = 0.661\ m,\ a_2 = 0.073\ m,\ a_3 = 0.0098\ m, \quad m = 0.1\ kg,\ \delta = 0.1, \quad J = -0.007\ kg \cdot m^2.$$

These parameters correspond to the model of a stone having the shape of an ellipsoid, and all obtained results of numerical experiments coincide with full-scale experiments.

The region of stability of permanent rotations for this model is presented at Fig. 4. Numerical experiments

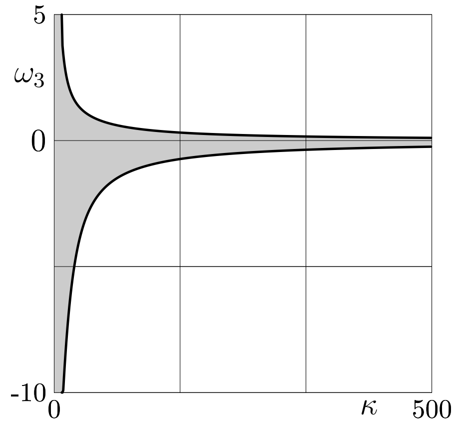

Fig. 4

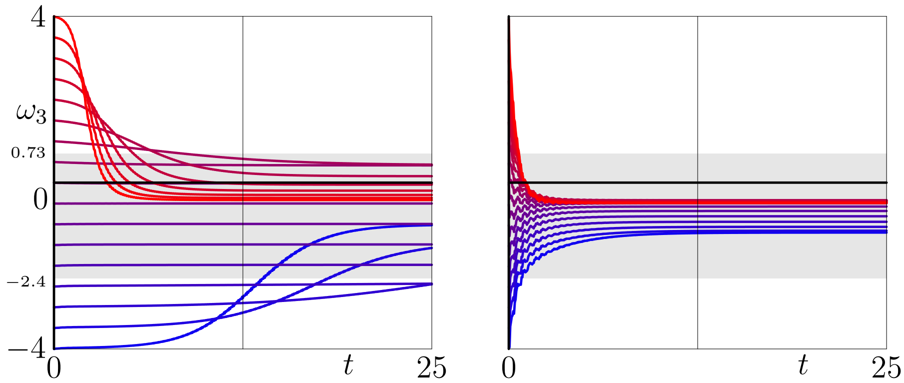

Fig. 5: $\gamma_3(0) = 0.99999$ and $\gamma_3(0) = 0.999$

were carried out on a plane with a coefficient of friction $k = 50\ s^{-1}$. The initial conditions had the form

$$\gamma_2(0) = 0, \quad \omega_1(0) = \omega_2(0) = 0, \ \mathbf{u}(0) = 0. \tag{10}$$

At Fig. 5 on the left are the results of experiments with different initial angular velocities of rotations with a very small initial deviation from the vertical ($\gamma_3(0) = 0.99999$), on the right for greater deviation ($\gamma_3(0) = 0.999$). The area of stability of rotations with a selected coefficient of viscous friction is highlighted in gray. As we see, stable rotations with a positive angular velocity of rotation have a very small region of attraction.

We note that in the non-holonomic statement of the problem for the chosen parameters there are no stable rotations ($J < 0$). In the case of the plane with friction and a small positive initial angular velocity of rotation, there is a change in the direction of rotation of the stone, with a subsequent exit to the stable uniform rotations.

The results of numerical experiments with sufficiently large initial angular velocities are presented at Fig. 6. Here such property of a stone, as the transition of rotational motions to vibrational motions and vice versa is observed. The final movement with a negative initial speed can be rotation in both the negative and positive directions.

5 Transient processes

For study the transient processes shown at Fig. 5 the equations (1)–(3) up to second-order terms in the variables $v_1, v_2, \omega_1, \omega_2, \gamma_1, \gamma_2$ are considered.

$$\dot{\mathbf{x}} = A(k, \delta, \omega_3)\tilde{\mathbf{x}} + \mathbf{b}(\mathbf{x}, k, \delta, \omega_3) + O_3(\widetilde{\mathbf{x}}) \tag{11}$$

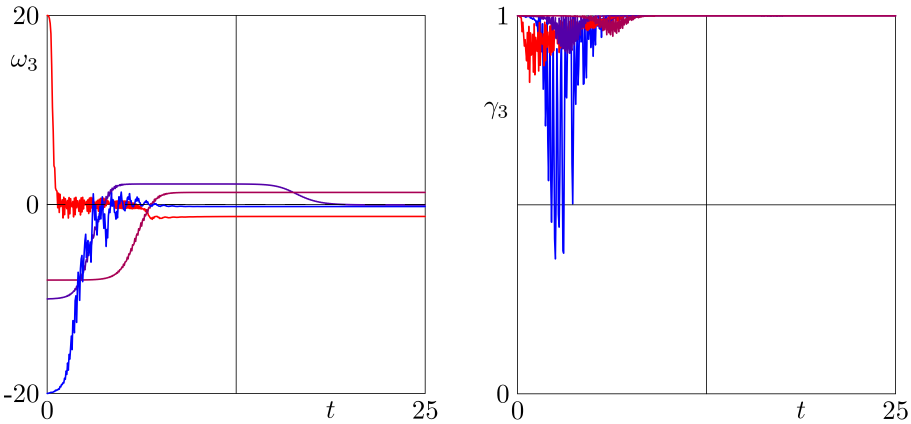

Fig. 6

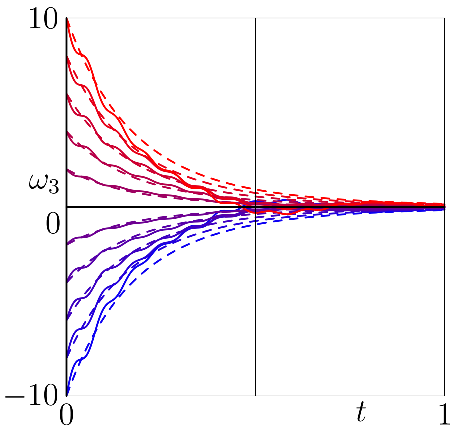

Fig. 7: $\delta = 0, \quad \gamma_3(0) = 0.999$

The changing of variables

$$v_1 = v\cos\psi, \quad v_2 = v\sin\psi, \quad \omega_1 = \rho_2\cos\varphi_2, \quad \omega_2 = \rho_1\sin\varphi_1, \quad \gamma_1 = \frac{\rho_1}{\xi_1}\cos\varphi_1, \quad \gamma_2 = \frac{\rho_2}{\xi_2}\sin\varphi_2$$

is performed, where $\xi_{1,2}^2 = mg(a_{1,2} - a_3)/A_{2,1}$. Averaging over variables ψ, φ_1 and φ_2 is carried out. Then the equations Eq. (11) have a view

$$\begin{gathered} \dot{v} = -kv, \quad \dot{\rho}_1 = -k\frac{ma_3^2}{2A_2}\rho_1, \quad \dot{\rho}_2 = -k\frac{ma_3^2}{2A_1}\rho_2, \\ \dot{\omega}_3 = -k\frac{m}{2A_3}\left(\frac{a_1^2\cos^2\delta - a_2^2\sin^2\delta}{\xi_1^2}\rho_1^2 + \frac{a_2^2\cos^2\delta - a_1^2\sin^2\delta}{\xi_2^2}\rho_2^2\right)\omega_3 + \frac{mg}{2A_3}\sin 2\delta(a_1 - a_2)\left(\frac{\rho_1^2}{\xi_1^2} - \frac{\rho_2^2}{\xi_2^2}\right) \end{gathered} \tag{12}$$

For the case of the coincidence of the directions of the principal axes of the body with the directions of the principal curvatures ($\delta = 0$) the numerical solutions of the averaged system Eq. (12) (Fig. 7, dotted curves) and leanerized system (Fig. 7, solid curves) have a good coincidence. Solving the averaged system Eq. (12) for example, under the initial conditions Eq. (10), we have $v \equiv 0$, $\rho_1 = \rho_1(0)\exp(-mka_3^2t/(2A_2))$, $\rho_2 \equiv 0$ and equality

$$\omega_3 = \omega_3(0)e^{\alpha_1(\rho_1^2 - \rho_1(0)^2)}, \quad \alpha_1 = \frac{a_1^2A_2}{2a_3^2A_3\xi_1^2} \tag{13}$$

is fair. Under initial conditions differing only in the direction of rotation, the motions are completely analogous and differ only in sign ω_3, and the corresponding curves $\omega_3(t)$ Fig. 7 are symmetrical with respect to the horizontal.

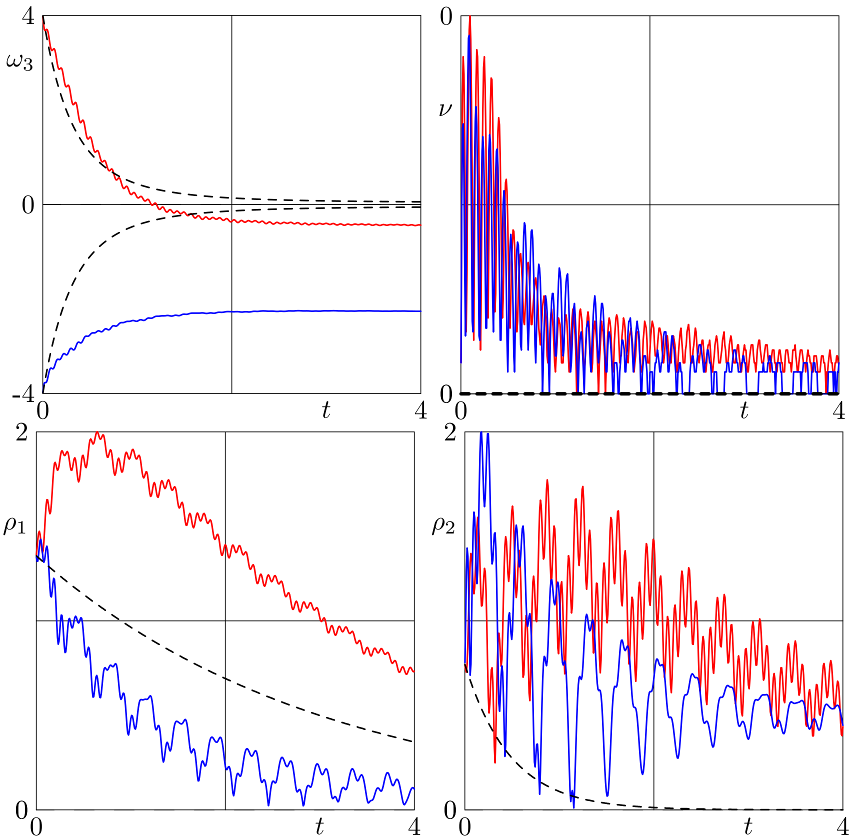

Fig. 8

This property is not preserved for the Celt ($\delta \neq 0$), and the equality Eq. (13) takes the form

$$\omega_3 = \omega_3(0)e^{\alpha_1(\rho_1^2 - \rho_1(0)^2)} + \frac{g(a_1 - a_2)\sin 2\delta}{2k(a_1^2\cos^2\delta + a_2^2\sin^2\delta)}\left(1 - e^{\alpha_1(\rho_1^2 - \rho_1(0)^2)(a_1^2\cos^2\delta + a_2\sin^2\delta)/a_1^2}\right) \tag{14}$$

In this case, the solutions of the averaged system, differing only in the initial direction of rotation, are not symmetric with respect to the horizontal. However, the displacements arising in this case are sufficiently small, and, depending on the initial conditions, they can be directed both to the lower and upper half-planes. In this case, the solutions of the system Eq. (11) deviate significantly from the solutions of the averaged system (Fig. 8), but in reality the displacement is directed to the lower half-plane. The changing in the direction of rotation of the stone is explained by the deviations of the exact solution from the averaged solution, and the final value of the angular velocity of rotation is always in a small neighborhood of zero.

6 Conclusions

Thus, the results of modeling the interaction of the Celt with the supporting plane by the force of viscous friction are consistent with the known properties of its dynamics, and it makes sense to investigate the considered problem further.

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