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Superelements in a minimal coordinates floating frame of reference formulation

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ABSTRACT — *In this work, the details of a new method for creating superelements for the dynamic analysis of flexible multibody systems is discussed. This method is based on the floating frame of reference formulation, but uses the absolute interface coordinates as degrees of freedom. In this way, it is possible to obtain a formulation in a minimal set of coordinates that does not require Lagrange multipliers to satisfy constraints. In the derivation of a flexible body's equation of motion, Craig-Bampton modes are used for describing the body's local elastic displacement field. The fact that these Craig-Bampton modes are able to describe rigid body motion is exploited to establish a coordinate transformation from the floating frame formulation's degrees of freedom to absolute interface coordinates. In earlier works, the details of the new formulation [1] and the details of the constraints imposed on the Craig-Bampton modes [2] were introduced. In this work, a convenient overview of the mathematical elaborations of both works is presented, as well as additional physical interpretations of the kinematic transformations involved in the formulation.*

1 Introduction

Flexible multibody dynamics is concerned with the study of machines and mechanisms that consist of multiple deformable bodies. These bodies are connected to each other or to the fixed world in their interface points. The joints that are located at these interface points may allow for large relative rigid body rotations between the bodies, which causes the problem to be of a geometrically nonlinear nature. However, the elastic strains and deformations within a single body can often be considered as small. In general, the methods suitable for the simulation of such flexible multibody systems can be classified in three formulations: the inertial frame formulations, the corotational frame formulations and the floating frame formulations. Between these formulations, essential differences can be found in the way the kinematics of a flexible body is described and in the way kinematic constraints between bodies are enforced.

The inertial frame formulation is based on the nonlinear Green-Lagrange strain definition. Each body is discretized in finite elements using global interpolation functions. Hence, the absolute nodal coordinates are used as degrees of freedom. As long as a body's interface points coincide with finite element nodes, constraints between bodies can be enforced directly, by equating the appropriate degrees of freedom of the nodes shared by both bodies. Due to the use of the nonlinear strain definition, no distinction is made between a body's large rigid body motion and small elastic deformation. Figure 1 shows a graphical representation of the inertial frame formulation.

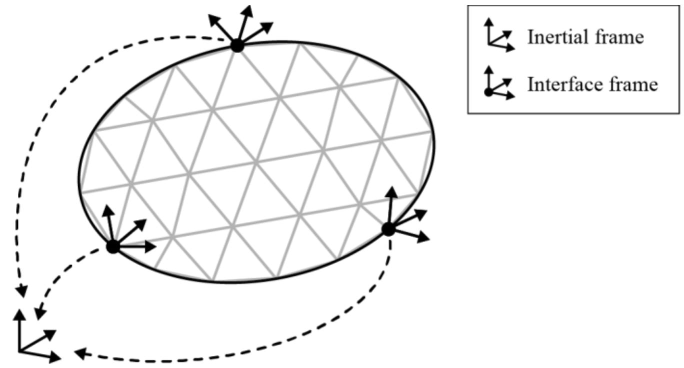

Fig. 1: Inertial frame formulation
The absolute interface coordinates are part of the degrees of freedom.

The corotational frame formulation can be interpreted as the nonlinear extension of the standard linear finite element formulation. Each element is given a corotational frame that describes the large rigid body motion of the element with the respect to the inertial frame. Small elastic deformations within the element are superimposed using the linear finite element matrices, based on the Cauchy strain definition. The nonlinear finite element model is obtained from the linear finite element model by pre- and post-multiplying the mass and stiffness matrices with the rotation matrices corresponding to the corotational frames. The absolute nodal coordinates are used as degrees of freedom, such that constraints are satisfied similarly as in inertial frame formulations. At every iteration, the orientation of the corotational frames are determined from the absolute nodal coordinates. Figure 2 shows a graphical representation of the corotational frame formulation.

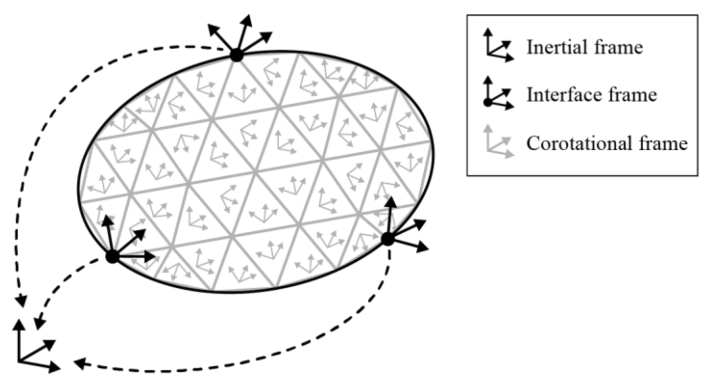

Fig. 2: Corotational frame formulation
The absolute interface coordinates are part of the degrees of freedom.

The floating frame formulation can be interpreted as the extension of rigid multibody formulations to flexible multibody systems. In this formulation, a body's large rigid body motion is described by the absolute coordinates of a floating frame that moves along with the body. Elastic deformation is described locally, relative to the floating frame using a linear combination of mode shapes. Within the framework of linear elasticity theory, these mode shapes can be determined from a body's linear finite element model. To this end, powerful model order reduction techniques such as the Craig-Bampton method can be used [3]. The degrees of freedom consist of the floating frame coordinates and the generalized coordinates corresponding to the flexible modes. As a consequence of the fact that the absolute interface coordinates are not part of the degrees of freedom, the kinematic constraint equations are nonlinear and in general difficult to solve analytically. Hence, Lagrange multipliers are required to satisfy the constraint equations when formulating the equations of motion. This increases the total number of unknowns in the constrained equations of motion and makes them of the differential-algebraic type. Figure 3 shows a graphical representation of the floating frame formulation.

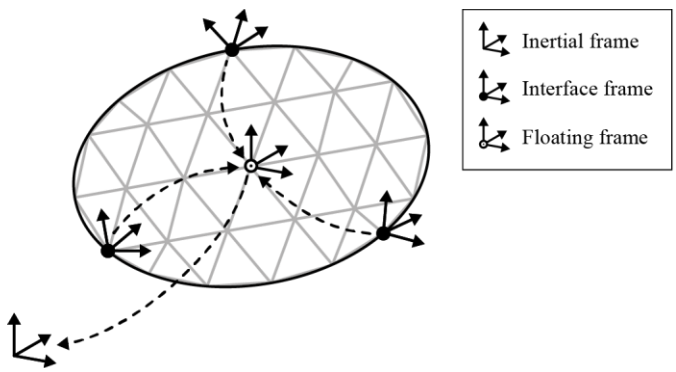

Fig. 3: Floating frame formulation

The absolute floating frame coordinates and the local interface coordinates are the degrees of freedom.

The fact that the floating frame formulation is able to exploit the advantages of linear model order reduction techniques, makes it a very efficient formulation for cases in which the elastic deformation of bodies can be considered as small. In order to develop a more efficient formulation, it is desired to combine this advantage with the convenient way in which the inertial frame formulation and corotational formulation satisfy the kinematic constrains. To this end, the Lagrange multipliers need to be eliminated from the floating frame formulation. This can be done if the absolute interface coordinates uniquely describe the body's kinematics. In other words, if it is possible to express both the floating frame coordinates and the generalized coordinates corresponding to local elastic mode shapes in terms of the absolute interface coordinates, the Lagrange multipliers can be eliminated. One could say that in this case a so-called superelement is created: the motion of a flexible body is described entirely by the motion of its interface points [4,5]. The term superelement refers to the similar way in which the displacement field of a finite element is described uniquely by the displacements of its nodes. In this sense, the problem of relating an element's corotational frame to its absolute nodal coordinates as for instance addressed in [6] is similar to the problem of relating a body's floating frame to its absolute interface coordinates.

In [1], the authors have presented a new method for creating superelements based on the floating frame formulation. In this method, the static Craig-Bampton modes (also referred to as interface modes) are used to describe a body's elastic deformation. Essential for the method is the fact that these Craig-Bampton modes are able to describe rigid body motions. In order to describe the system's motion uniquely, these rigid body modes must be eliminated. In [1], it was shown that this property can be used to establish a coordinate transformation that expresses both the floating frame coordinates and the local interface coordinates corresponding to the Craig-Bampton modes in terms of the absolute interface coordinates. This is done by demanding that the elastic body has no deformation at the location of the floating frame, which results in an unique definition of the floating frame. Hence, the strength of the new method as presented in [1] is that it removes the rigid body motion from the Craig-Bampton modes and relates the floating frame coordinates uniquely to the absolute interface coordinates simultaneously.

In this work, an overview will be presented of this new absolute interface coordinates floating frame formulation with which superelements can be created efficiently. Recent submitted work [2], includes a convenient mathematical overview of the constraints that should be imposed on the Craig-Bampton modes in order to remove the rigid body modes. In this work, the mathematical form of these constraints is used to explain the fundamental differences between the new method and previously published alternatives.

The subsequent sections are organized as follows: In Sect. 2, the constraints imposed on the Craig-Bampton modes are presented. This is used to establish the desired coordinate transformation towards absolute interface coordinates. In this section, interpretations will be given to the way the new method satisfies these constraints, compared to alternatives reported in literature. In Sect. 3, the required kinematics of the floating frame formulation is introduced. The local interface coordinates, which describe the elastic deformation, are expressed in terms of the difference between the absolute interface coordinates and the absolute floating frame coordinates. Upon substitution in the constraint for the Craig-Bampton modes, removing the rigid body motion, the floating frame coordinates and local interface coordinates are both expressed in terms of the absolute interface coordinates. In Sect. 4, a summary of the derivation of the equation of motion of a flexible body in terms of the absolute interface coordinates is presented. This derivation is based on the equation of motion of the standard floating frame formulation. In this section, important comments will be made with regards to the numerical solution procedure.

2 Local kinematics of a flexible body using Craig-Bampton modes

As explained in the introduction, a coordinate transformation from the absolute floating frame coordinates and generalized coordinates corresponding to the local elastic mode shapes to the absolute interface coordinates is required. The fact that this coordinate transformation involves the interface coordinates, suggests naturally to use the Craig-Bampton modes to describe the body's local elastic behavior. After all, the generalized coordinates corresponding to the Craig-Bampton modes are the local interface coordinates.

Consider a three-dimensional flexible body of which the position and orientation of the floating frame is denoted by the pair $\{P_j, E_j\}$. In this, P_j identifies the material point on the body to which coordinate frame E_j is rigidly attached. Let N be the number of interface points on a flexible body. Then the number of interface coordinates and thus the number of Craig-Bampton modes is $6N$. Let P_k identify the interface point with index k. The local interface coordinates corresponding to this point are denoted by the (6×1) vector $\mathbf{q}_k^{j,j}$, which defines the elastic displacements and rotations of P_k (lower index k) relative to P_j (second upper index j) and its components are expressed in the coordinate system $\{P_j, E_j\}$ (first upper index j). Now, the local elastic deformation of an arbitrary point P_A on the body can be expressed in terms of the local interface coordinates as:

$$\mathbf{q}_A^{j,j} = \sum_{k=1}^{N} \mathbf{\Phi}_k\left(\mathbf{x}_A^{j,j}\right) \mathbf{q}_k^{j,j}. \tag{2.1}$$

Here $\mathbf{\Phi}_k$ is the (6×6) matrix of Craig-Bampton modes of P_k. It describes the local elastic displacements and rotations of the material point P_A, which has position $\mathbf{x}_A^{j,j}$ on the undeformed body. Equation (2.1) can be written in compact matrix-vector form as:

$$\mathbf{q}_A^{j,j} = [\mathbf{\Phi}_A]\mathbf{q}^{j,j}, \tag{2.2}$$

where $[\mathbf{\Phi}_A]$ is the $(6 \times 6N)$ matrix of Craig-Bampton modes evaluated at P_A and the $(1 \times 6N)$ vector $\mathbf{q}^{j,j}$ contains all local interface coordinates:

$$[\mathbf{\Phi}_A] \equiv \left[\mathbf{\Phi}_1\left(\mathbf{x}_A^{j,j}\right) \quad \dots \quad \mathbf{\Phi}_N\left(\mathbf{x}_A^{j,j}\right)\right], \qquad \mathbf{q}^{j,j} \equiv \begin{bmatrix} \mathbf{q}_1^{j,j} \\ \vdots \\ \mathbf{q}_N^{j,j} \end{bmatrix}, \tag{2.3}$$

When using the Craig-Bampton modes above in the floating frame formulation, the total number of degrees of freedom will be $6 + 6N$: there are 6 absolute floating frame coordinates and $6N$ local interface coordinates. However, the Craig-Bampton modes are still able to describe 6 rigid body motions, which causes problems of non-uniqueness because the rigid body motion is already described by the floating frame coordinates. Moreover, a formulation is desired in terms of the $6N$ absolute interface coordinates. Hence, in order to establish a unique coordinate transformation, 6 constraints should be imposed on the local interface coordinates $\mathbf{q}^{j,j}$, corresponding to the Craig-Bampton modes:

$$\boldsymbol{\mathcal{F}}\left(\mathbf{q}^{j,j}\right) = \mathbf{0}. \tag{2.4}$$

Due to the possible nonlinearities in the general form of Eq. (2.4), these constraints might not be solved analytically. However, by taking its variation, 6 linear equations are obtained in terms of the virtual displacements of the interface points $\delta\mathbf{q}^{j,j}$:

$$\nabla\boldsymbol{\mathcal{F}} \cdot \delta\mathbf{q}^{j,j} = \mathbf{0}. \tag{2.5}$$

If Eq. (2.5) is satisfied, the virtual displacements of the interface points are such that no rigid body motion occurs. In [1] this is realized by defining the floating frame such that there is zero elastic deformation at its location. By taking the variation of Eq. (2.2) and evaluating the Craig-Bampton modes at the location of the floating frame P_j, a constraint is obtained of the form of Eq. (2.5):

$$\left[\boldsymbol{\Phi}_j\right]\delta\mathbf{q}^{j,j} = \mathbf{0}. \tag{2.6}$$

There are multiple ways in which Eq. (2.6) can be satisfied. It is clear that when $\left[\boldsymbol{\Phi}_j\right]$ itself equals zero, the constraint is always satisfied. This means that each individual Craig-Bampton mode equals zero at the location of the floating frame. A first possibility for which this is true, is when the floating frame is located at an interface point, and the Craig-Bampton modes of that specific interface point are not taken into account [4]. Figure 4 shows a graphical representation of this situation. An important disadvantage of this method is that simulation results become dependent on which interface point is chosen. Moreover, a better accuracy can be expected when the floating frame is located close to the body's center of mass.

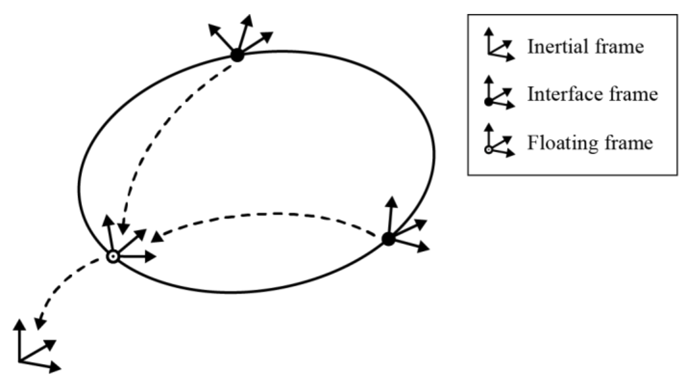

Fig. 4: Floating frame is located in an interface point.

A possibility for which $\left[\boldsymbol{\Phi}_j\right]$ equals zero and with which the floating frame can be positioned close to the center of mass, is to add an auxiliary interface point at the material point that coincides with the center of mass of the undeformed body. The Craig-Bampton modes are then determined while keeping this auxiliary interface point fixed [5]. Figure 5 shows a graphical representation of this situation. In comparison to the first method, the accuracy of the second method is in general better, but it also has 6 additional degrees of freedom. Moreover, it is required to determine the location of the floating frame before computing the Craig-Bampton modes. Consequently, if one wants to locate the floating frame on a different location, these modes need to be recomputed.

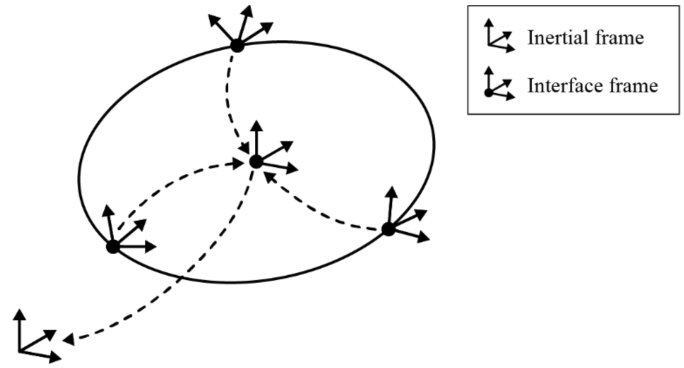

Fig. 5: Floating frame is treated as an auxiliary interface point at the center of mass of the underformed body.

The method presented in [1] allows to locate the floating frame at the center of mass of the undeformed body, without using an auxiliary interface point. Hence, it does not introduce 6 additional degrees of freedom. Figure 6 shows a graphical representation of this situation. In order to arrive at this formulation, it does not impose constraints on the Craig-Bampton modes individually. As long as any linear combination of the modes is zero at the location of the floating frame, Eq. (2.6) is satisfied. In the next chapter, it will be shown how the location of the floating frame follows from satisfying these constraints.

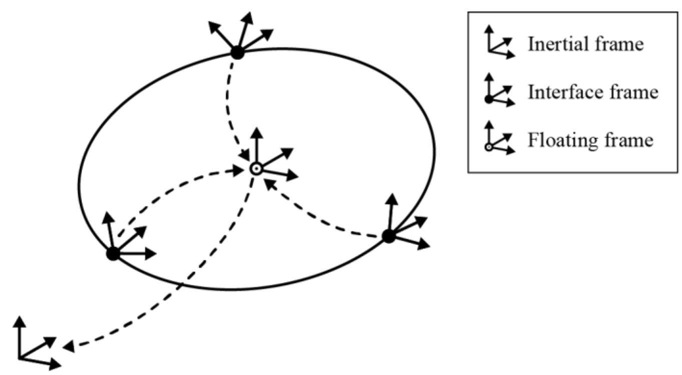

Fig. 6: Floating frame is located at the center of mass of the underformed body, which is not an interface point.

3 Coordinate transformation to absolute interface coordinates

Recall Fig. 3 in which it is shown that the absolute interface coordinates are expressed in terms of the absolute floating frame coordinates and the local interface coordinates. For the position vector of interface point P_k with respect to the inertial frame P_o, with its components expressed in the inertial frame P_o holds:

$$\mathbf{r}_k^{O,O} = \mathbf{r}_j^{O,O} + \mathbf{R}_j^O \mathbf{r}_k^{j,j}, \tag{3.1}$$

where the indices of the position vectors follow the notation convention as introduced in the previous section and $\mathbf{R}_j^O$ is the (3×3) rotation matrix that relates the orientation of E_j to E_O. In order to obtain an expression for the virtual displacement of interface point P_k, the variation of Eq. (3.1) is taken. This requires the variation of the rotation matrix $\delta\mathbf{R}_j^O$. Based on the orthogonality property of the rotation matrix, it can be shown that the variation can be written as:

$$\delta\mathbf{R}_j^O = \delta\tilde{\boldsymbol{\pi}}_j^{O,O}\mathbf{R}_j^O. \tag{3.2}$$

Here, $\delta\tilde{\boldsymbol{\pi}}_j^{O,O}$ is a skew symmetric matrix of virtual rotations, containing the components of the virtual rotation vector. With Eq. (3.2), the variation of Eq. (3.1) is obtained:

$$\delta\mathbf{r}_k^{O,O} = \delta\mathbf{r}_j^{O,O} + \delta\tilde{\boldsymbol{\pi}}_j^{O,O}\mathbf{R}_j^O\mathbf{r}_k^{j,j} + \mathbf{R}_j^O\delta\mathbf{r}_k^{j,j}. \tag{3.3}$$

For the virtual rotation of interface point P_k holds:

$$\delta\boldsymbol{\pi}_k^{O,O} = \delta\boldsymbol{\pi}_j^{O,O} + \mathbf{R}_j^O\delta\boldsymbol{\pi}_k^{j,j}. \tag{3.4}$$

Equations (3.3) and (3.4) can be combined in a compact notation, by introducing $\delta\mathbf{q}_k^{O,O}$ as the virtual change in the global interface coordinates of P_k, i.e. the combined set of virtual displacements $\delta\mathbf{r}_k^{O,O}$ and virtual rotations $\delta\boldsymbol{\pi}_k^{O,O}$:

$$\delta\mathbf{q}_k^{O,O} = \left[\mathbf{R}_j^O\right]\delta\mathbf{q}_k^{j,j} + \left[\mathbf{R}_j^O\right]\left[-\tilde{\mathbf{r}}_k^{j,j}\right]\left[\mathbf{R}_O^j\right]\delta\mathbf{q}_j^{O,O}, \tag{3.5}$$

in which

$$\left[\mathbf{R}_j^O\right] \equiv \begin{bmatrix} \mathbf{R}_j^O & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_j^O \end{bmatrix}, \qquad \left[-\tilde{\mathbf{r}}_k^{j,j}\right] \equiv \begin{bmatrix} \mathbf{1} & -\tilde{\mathbf{r}}_k^{j,j} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}. \tag{3.6}$$

It is interesting to note that the columns of the (6×6) matrix $\left[-\tilde{\mathbf{r}}_k^{j,j}\right]$ contain the displacements of interface point P_k when the deformed body with floating frame P_j is subjected to infinitesimal rigid body motions: the first three columns representing the displacements due to rigid body translations and the last three columns representing the displacements due to rigid body rotations.

Equation (3.5) can be established for all interface points:

$$\delta\mathbf{q}^{O,O} = \left[\bar{\mathbf{R}}_j^O\right]\delta\mathbf{q}^{j,j} + \left[\bar{\mathbf{R}}_j^O\right]\left[\boldsymbol{\Phi}_{rig}\right]\left[\mathbf{R}_O^j\right]\delta\mathbf{q}_j^{O,O}, \tag{3.5}$$

in which $\delta\mathbf{q}^{O,O}$ and $\delta\mathbf{q}^{j,j}$ are the $(6N \times 1)$ vectors containing all variations of the absolute and local interface coordinates respectively, $\left[\bar{\mathbf{R}}_j^O\right]$ is the $(6N \times 6N)$ block diagonal rotation matrix and $\left[\boldsymbol{\Phi}_{rig}\right]$ is the column-wise assembly of all matrices $\left[-\tilde{\mathbf{r}}_k^{j,j}\right]$ such that it represents the displacements all interface points, when the deformed body is given a rigid body motion:

$$\left[\bar{\mathbf{R}}_O^j\right] \equiv \begin{bmatrix} \left[\mathbf{R}_O^j\right] & & \\ & \ddots & \\ & & \left[\mathbf{R}_O^j\right] \end{bmatrix}, \qquad \left[\boldsymbol{\Phi}_{rig}\right] \equiv \begin{bmatrix} \left[-\tilde{\mathbf{r}}_1^{j,j}\right] \\ \vdots \\ \left[-\tilde{\mathbf{r}}_N^{j,j}\right] \end{bmatrix}. \tag{3.6}$$

Equation (3.5) can be rewritten such that the virtual change in the local interface coordinates is expressed as the difference between the absolute interface coordinates and the absolute floating frame coordinates. Figure 7 shows this graphically:

$$\delta\mathbf{q}^{j,j} = \left[\bar{\mathbf{R}}_O^j\right]\delta\mathbf{q}^{O,O} - \left[\boldsymbol{\Phi}_{rig}\right]\left[\mathbf{R}_O^j\right]\delta\mathbf{q}_j^{O,O}. \tag{3.7}$$

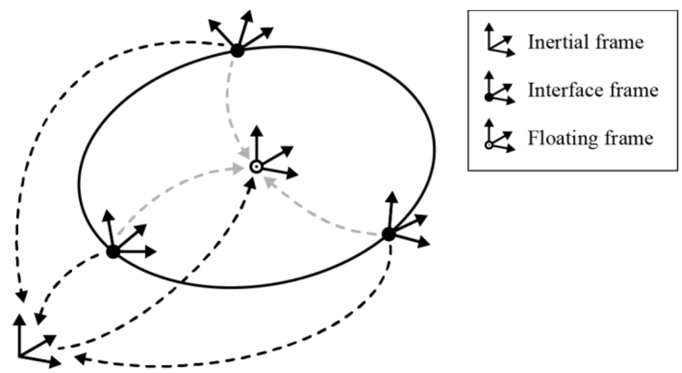

Fig. 7: The local interface coordinates (grey arrows) are expressed as the difference between the absolute interface coordinates and the absolute floating frame coordinates (black arrows).

Recall that the Craig-Bampton modes used in the above kinematic expressions are still able to describe rigid body motions. In order to remove these rigid body motion, the constraint Eq. (2.6) needs to be satisfied. At this point, Eq. (3.7) can be substituted in the constraint Eq. (2.6). This results in 6 equations from which the variations of the 6 floating frame coordinates can be expressed in terms of the absolute interface coordinates:

$$\delta \mathbf{q}_j^{O,O} = [\mathbf{R}_j^O]\left([\mathbf{\Phi}_j][\mathbf{\Phi}_{rig}]\right)^{-1}[\mathbf{\Phi}_j][\bar{\mathbf{R}}_O^j]\delta \mathbf{q}^{O,O}. \tag{3.8}$$

Or in short:

$$\delta \mathbf{q}_j^{O,O} = [\mathbf{R}_j^O][\mathbf{Z}_j][\bar{\mathbf{R}}_O^j]\delta \mathbf{q}^{O,O}, \qquad [\mathbf{Z}_j] \equiv \left([\mathbf{\Phi}_j]\,[\mathbf{\Phi}_{rig}]\right)^{-1}[\mathbf{\Phi}_j]. \tag{3.9}$$

This expression can be interpreted as follows: the absolute interface coordinates $\delta \mathbf{q}^{O,O}$ are transformed to the local frame by the compound rotation matrix $[\bar{\mathbf{R}}_O^j]$. The matrix $[\mathbf{Z}_j]$ determines the displacement of the floating frame when the interface points are given a unit displacement. For an undeformed body, this information is contained within the Craig-Bampton mode matrix $[\mathbf{\Phi}_j]$. The pre-multiplication with the inverse of $[\mathbf{\Phi}_j]\,[\mathbf{\Phi}_{rig}]$ can be seen as a correction for the fact that the body is able to deform. Finally, the rotation matrix $[\mathbf{R}_j^O]$ is used to obtain the floating frame coordinates in the inertial frame.

By back substitution of Eq. (3.9) in Eq. (3.7), it is also possible to express the variations in the local interface coordinates in terms of the variations in the global interface coordinates. In short form, this can be written as:

$$\delta \mathbf{q}^{j,j} = [\mathbf{T}_j][\bar{\mathbf{R}}_O^j]\delta \mathbf{q}^{O,O}, \qquad [\mathbf{T}_j] \equiv \mathbf{1} - [\mathbf{\Phi}_{rig}^j][\mathbf{Z}_j]. \tag{3.10}$$

This expression can be interpreted as follows: $[\bar{\mathbf{R}}_O^j]$ transforms the absolute interface coordinates to the local frame. Then, by means of the term $-[\mathbf{\Phi}_{rig}^j][\mathbf{Z}_j]$, the transformation matrix $[\mathbf{T}_j]$ removes the rigid body motion of the floating frame from the expression.

The combination of Eq. (3.9) and (3.10) can be used to express the degrees of freedom used in the standard floating frame formulation in terms of the absolute interface coordinates that are desired to satisfy the kinematic constraints without Lagrange multipliers. In compact form, these coordinate transformations can be written as:

$$\begin{bmatrix} \delta \mathbf{q}_j^{O,O} \\ \delta \mathbf{q}^{j,j} \end{bmatrix} = \mathbf{A}\delta \mathbf{q}^{O,O}, \qquad \mathbf{A} \equiv \begin{bmatrix} [\mathbf{R}_j^O][\mathbf{Z}^j][\bar{\mathbf{R}}_O^j] \\ [\mathbf{T}^j][\bar{\mathbf{R}}_O^j] \end{bmatrix}. \tag{3.11}$$

4 Equations of motion in absolute interface coordinates

The equations of motion in terms of absolute interface coordinates can be derived using the principle of virtual work. Following the standard floating frame formulation, the virtual position, velocity and acceleration of any arbitrary point on a flexible body are expressed in terms of the chosen degrees of freedom: the absolute floating frame coordinates and the local interface coordinates corresponding to the Craig-Bampton modes. At this point, the coordinate transformation in Eq. (3.11) could be substituted. Then, the inertia integral can be computed from which the mass matrix and fictitious forces due to quadratic velocity terms are obtained. However, since the equations of motion in the floating frame formulation are well-established, these can be used as a starting point too:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q}. \tag{4.1}$$

In this, $\mathbf{q}$ is the vector of generalized coordinates: it contains the absolute floating frame coordinates and the local interface coordinates. $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices according to the floating frame formulation. The term $\mathbf{C}\dot{\mathbf{q}}$ represents the fictitious inertia forces. $\mathbf{Q}$ contains the generalized forces that are externally applied or due to kinematic constraints.

The kinematic transformation Eq. (3.11) relates virtual displacements. Following the same procedure, it can be shown that the same transformation matrix $\mathbf{A}$, can be used on the level of velocities. Consequently, for the accelerations hold:

$$\begin{bmatrix} \ddot{\mathbf{q}}_j^{O,O} \\ \ddot{\mathbf{q}}^{j,j} \end{bmatrix} = \mathbf{A}\ddot{\mathbf{q}}^{O,O} + \dot{\mathbf{A}}\dot{\mathbf{q}}^{O,O}. \tag{4.2}$$

The coordinate transformation can be substituted in the equations of motion (4.1), which yields:

$$\mathbf{A}^T\mathbf{M}\mathbf{A}\ddot{\mathbf{q}}^{O,O} + \mathbf{A}^T\left(\mathbf{M}\dot{\mathbf{A}} + \mathbf{C}\mathbf{A}\right)\dot{\mathbf{q}}^{O,O} + \mathbf{A}^T\mathbf{K}\mathbf{q} = \mathbf{A}^T\mathbf{Q}. \tag{4.3}$$

Let $\mathbf{M}_{CB}$ and $\mathbf{K}_{CB}$ represent the local mass and stiffness matrices that are obtained from a body's local linear finite element matrices by model order reduction based on the Craig-Bampton modes. By carrying out the matrix multiplications in Eq. (4.3) directly, it is tedious but straightforward to show that the transformed mass and stiffness matrices are related to $\mathbf{M}_{CB}$ and $\mathbf{K}_{CB}$ as follows:

$$\left[\bar{\mathbf{R}}_j^O\right]\mathbf{M}_{CB}\left[\bar{\mathbf{R}}_O^j\right]\ddot{\mathbf{q}}^{O,O} + \mathbf{Q}_v + \left[\bar{\mathbf{R}}_j^O\right]\left[\mathbf{T}_j\right]^T\mathbf{K}_{CB}\mathbf{q}^{j,j} = \mathbf{Q}^O, \tag{4.4}$$

which is the equation of motion in terms of the absolute interface coordinates. In this sense, it can be interpreted as the equation of motion of the superelement. The transformed mass matrix is simply the reduced mass matrix rotated to the inertial frame. In the elastic term, the reduced stiffness matrix is multiplied by the local interface coordinates, resulting in the local elastic force vector. The pre-multiplication with $\left[\mathbf{T}_j\right]^T$ can be interpreted as an operation that eliminates the rigid body component. For an undeformed body, this would equal the identity matrix, but once in the deformed configuration this is not the case, see also the definition of $\left[\mathbf{T}_j\right]$ in equation (3.10). The resulting elastic forces are then rotated to the global frame. Hence, the mass and stiffness matrix for the superelement can be conveniently obtained from the reduced mass and stiffness matrices.

Unfortunately the fictitious forces in Eq. (4.4), represented by $\mathbf{Q}_v$, cannot be related directly to the reduced mass matrix. In order to be exact, the matrix multiplication as in Eq. (4.3) needs to be computed. Only when additional assumptions are made, this expression can be simplified. This can be done for instance by lumping the reduced mass matrix and will in general result in accurate simulation results. The externally applied forces and constraint forces are represent by $\mathbf{Q}^O$, which contains in fact the generalized forces at the interface points, expressed in the inertial frame. Because the kinematic constraints are enforced at the interface points, the constraint forces will simply drop out $\mathbf{Q}^O$ as soon as the equation of motion of the entire multibody system is formulated.

The fact that it is not possible to establish a coordinate transformation on the position level, is the reason that the elastic forces in Eq. (4.4) are still expressed in terms of the local instead of the absolute interface coordinates. For this reason, some remarks about the procedure with which the equation of motion (4.4) are solved numerically are appropriate. To this end, it is important to realize that the equations of motion will be solved numerically indeed. This means that the equations of motion is not solved for the large absolute position of the interface points. Instead, it is only solved for the small increment in the interface coordinates that occurs during the time increment. The time-discretized equations are linear in this position increment and tangent to the current configuration space. Consequently, the transformation matrix in Eq. (3.11) can be used to ensure that at every time increment, the increment in the absolute interface coordinates is solved.

At the next time increment, it is required to determine the floating frame coordinates and local interface coordinates such that the rotation matrix $\mathbf{R}_j^O$ and the elastic forces can be computed. With the increment in the absolute interface coordinates known, the increment for the floating frame coordinates can be determined from Eq. (3.9). However, the error introduced by numerical integration may cause the floating frame to drift. For that reason Newton-Raphson iterations can be applied in which the current floating frame coordinates are used as an initial estimate. In simulations that were performed for validation purposes in [1] and [2], it was found that only few Newton-Raphson iterations are actually required. Once both the absolute interface coordinates and the absolute floating frame coordinates are determined with sufficient accuracy, the local interface coordinates and thus the local elastic deformation can be determined directly. At this point, the equations of motion can be solved again.

In both [1] and [2] a wide variety of benchmark problems was simulated using the new method discussed in this work. Both static and dynamic simulations were performed and the results obtained with the new method were compared with the inertial frame formulation of ANSYS, the floating frame formulation of ADAMS and the corotational formulation of SPACAR [7]. These simulations contain: the equilibrium analysis of a cantilever beam that is subjected to a large tip force, the dynamic analysis of a cantilever beam subjected to a transient tip force, the dynamic analysis of systems of high-speed rotating beams and the dynamic analysis of 2D and 3D slider-crank mechanisms. All simulations have shown that the new method produces accurate results. Moreover, it was shown that when using a small number of bodies, it outperforms the traditional floating frame formulation in which the floating frame was located in an interface point.

5 Conclusions

In this work, the characteristics of three fundamentally different flexible multibody formulations were discussed with regards to the chosen degrees of freedom and the way in which kinematic constraint equations are enforced. Multibody systems of which the elastic deformations of individual bodies remain sufficiently small can be studied efficiently using the floating frame formulation. This is due to the fact that in this formulation, elastic deformations are described locally. As a consequence powerful model order reduction methods can be applied on a body's linear finite element model to obtain the local reduced mass and stiffness matrices. The main disadvantage of the traditional floating frame formulation is that it requires Lagrange multipliers to satisfy kinematic constraints, because the absolute coordinates of the interface points, where the constraints are applied, are not part of the degrees of freedom.

It was explained that the Lagrange multipliers can be eliminated from the formulation, when a coordinate transformation from the floating frame formulation's degrees of freedom to absolute interface coordinates is established. This can in fact be interpreted as creating a superelement from a body's floating frame formulation. The central role of the interface coordinates in this matter, suggests that using Craig-Bampton modes for the local description of a body's displacement field is convenient, because the local interface coordinates equal the generalized coordinates corresponding to these Craig-Bampton modes.

An important property of the Craig-Bampton modes is that they are able to describe rigid body motions. In order to describe the kinematics uniquely, these rigid body modes must be eliminated. It was explained in detail which 6 constraints can be imposed on the local interface coordinates. Traditional methods have satisfied these constraints by locating the floating frame in an (auxiliary) interface point. This means that these methods demand that each Craig-Bampton modes equal zero at the location of the floating frame. The newly presented method only demands that any linear combination of the Craig-Bampton modes produces zero elastic deformation at the location of the floating frame. In this way, the new method is able to locate the floating frame conveniently in the center of mass of the undeformed body, without the need of an auxiliary interface point (and thus 6 additional degrees of freedom) there.

By satisfying the 6 constraints imposed on the Craig-Bampton modes, the coordinate transformations from the absolute floating frame coordinates and local interface coordinates to absolute interface coordinates was obtained. Upon substitution in the floating frame formulation's equations of motion of a flexible body, the equation of motion of the superelement is obtained. It was explained that the final mass and stiffness matrices can be obtained directly from a body's reduced linear finite element model. Important aspects with regards to the numerical solution procedure were outlined. In this way, a clear overview of the new method of creating superelements, as discussed in [1] and [2] is presented.

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