Modified Newmark formulas for the rotational equations of motion of a rigid body when using Euler parameters

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The most common and widely used type of rotational coordinates in multibody systems that treat the orientation are Euler angles. An alternative non-minimal representation of the orientation of a body is given by using four Euler parameters. The rotational equations of motion for a rigid body in terms of Euler parameters $\mathbf{p}$ is given by the following differential-algebraic system [1, 2]

$$4\mathbf{L}^T\mathbf{J}\mathbf{L}\ddot{\mathbf{p}} + 8\mathbf{L}^T\mathbf{L}\dot{\mathbf{L}}^T\mathbf{J}\mathbf{L}\dot{\mathbf{p}} + \mathbf{p}\lambda = 2\mathbf{L}^T\overline{\mathbf{m}} \tag{1}$$

$$\mathbf{p}^T\mathbf{p} - 1 = 0 \tag{2}$$

The algebraic equation (2) results from the fact that four Euler parameters are over-determined for the three rotational degrees of freedom of the body in space.

The present paper discusses the numerical damping effect when solving the latter equations using the well-known Newmark formulas for the discretization. The classical Newmark formulas are given by [3]

$$\mathbf{p}_{n+1} = \mathbf{p}_n + h\dot{\mathbf{p}}_n + \frac{h^2}{2}(1-2\beta)\ddot{\mathbf{p}}_n + h^2\beta\ddot{\mathbf{p}}_{n+1} \tag{3}$$

$$\dot{\mathbf{p}}_{n+1} = \dot{\mathbf{p}}_n + h(1-\gamma)\ddot{\mathbf{p}}_n + h\gamma\ddot{\mathbf{p}}_{n+1} \tag{4}$$

in which all quantities with subscript n are given at the time step t_n, and $n+1$ corresponds to time step t_{n+1} with $t_{n+1} = t_n + h$. The parameters β in Eq. (3) and γ in Eq. (4) can be computed by the damping parameter α as

$$\beta = \frac{1}{4}(1-\alpha)^2 \quad \text{and} \quad \gamma = \frac{1}{2}(1-2\alpha). \tag{5}$$

The necessary suggestion for $\ddot{\mathbf{p}}_{n+1}$ is computed by the Hilber, Hughes and Taylor (HHT) α-method [4], which includes a variable damping depending on the value of the parameter α in the interval $[-\frac{1}{3}, 0]$. The smaller the value of α the larger the artificial damping can be chosen in order to damp out high frequency oscillations that are of no interest. When choosing $\alpha = 0$, there is no damping. However, it can be shown that even for very simple test cases without high frequency oscillations, energy dissipation occurs when using Euler parameters for the orientation of the body. Two test cases are discussed in order to demonstrate this undesirable effect of numerical damping.

In the first test case, an unconstraint rigid body rotates with a constant angular velocity ω about one axis. It can be shown that after one time step h the angular velocity of the body results in

$$\omega_{n+1} = \omega_n + \frac{1}{4}h^2\alpha\omega_n^3 + O[h]^4 \tag{6}$$

For $\alpha = 0$ no energy dissipation exists. Although in this simple case no high frequencies are present, a considerably damping effect can be identified when setting α unequal to zero.

As second test case, a constant torque m_x about the x-axis is applied to an unconstraint rigid body which is at rest. The inertia moment of the body about the x-axis is defined as I_x. After one time step h, the angular velocity is here given by

$$\omega_{n+1} = \frac{m_x}{I_x}h - \frac{(1+\alpha+2\alpha^2)m_x^3}{64I_x^3}h^5 + O[h]^6 \tag{7}$$

For this test case, numerical damping always occurs even when $\alpha = 0$, which is a very astonishing result. At this point it should be mentioned again that the intension of setting $\alpha < 0$ is to damp out high frequencies of the mechanical model, and when setting α to zero no damping should occur. Both test cases show that the combination of the classical Newmark formulas with the Euler parameter description of the orientation of the body is an unfavorable choice.

Modified Newmark formulas are suggested in the present paper in order to get rid of this problem. The modification leads to a limitation of the energy dissipation for $\alpha < 0$, and reduces it to zero for $\alpha = 0$. While the suggestion of $\mathbf{p}_{n+1}$ is kept as in case of the classical Newmark formulas, i.e. Eq. (3), the suggestion of $\dot{\mathbf{p}}_{n+1}$ will be significantly changed. Instead of a discretization of the Euler parameter velocities $\dot{\mathbf{p}}_{n+1}$ as in Eq. (4), a discretization is applied to the angular velocity vector of the body. The relation between the angular velocity vector $\boldsymbol{\omega}$, the angular acceleration vector $\dot{\boldsymbol{\omega}}$ and the first and second Euler parameter time derivatives can be written as in [1, 2]

$$\boldsymbol{\omega} = 2\mathbf{L}\dot{\mathbf{p}} \qquad , \qquad \dot{\boldsymbol{\omega}} = 2\mathbf{L}\ddot{\mathbf{p}} \tag{8}$$

where the 3x4 matrix $\mathbf{L}$ depends on the Euler parameters. Thus, Eq. (4) is replaced with

$$\boldsymbol{\omega}_{n+1} = \boldsymbol{\omega}_n + h(1-\gamma)\dot{\boldsymbol{\omega}}_n + h\gamma\dot{\boldsymbol{\omega}}_{n+1} \tag{9}$$

yielding

$$2\mathbf{L}_{n+1}\dot{\mathbf{p}}_{n+1} = 2\mathbf{L}_n\dot{\mathbf{p}}_n + 2h(1-\gamma)\mathbf{L}_n\ddot{\mathbf{p}}_n + 2h\gamma\mathbf{L}_{n+1}\ddot{\mathbf{p}}_{n+1} \tag{10}$$

by using the relationships of Eq. (8)

Now, we just have three equations for the four parameters of the vector $\dot{\mathbf{p}}_{n+1}$. By differentiating the kinematic Euler parameter constraint, i.e. Eq. (2), with respect to time we obtain $2\mathbf{p}^T\dot{\mathbf{p}} = 0$. Extending the three equation of Eq. (10) with this relation we get

$$\begin{bmatrix} 2\mathbf{L}_{n+1} \\ 2\mathbf{p}_{n+1}^T \end{bmatrix} \dot{\mathbf{p}}_{n+1} = \begin{bmatrix} 2\mathbf{L}_n\dot{\mathbf{p}}_n + 2h(1-\gamma)\mathbf{L}_n\ddot{\mathbf{p}}_n + 2h\gamma\mathbf{L}_{n+1}\ddot{\mathbf{p}}_{n+1} \\ 0 \end{bmatrix} \tag{11}$$

Taking the inverse of the matrix on the left hand side we can write the new suggestion for the Euler parameter velocities

$$\dot{\mathbf{p}}_{n+1} = \begin{bmatrix} 2\mathbf{L}_{n+1} \\ 2\mathbf{p}_{n+1}^T \end{bmatrix}^{-1} \begin{bmatrix} 2\mathbf{L}_n\dot{\mathbf{p}}_n + 2h(1-\gamma)\mathbf{L}_n\ddot{\mathbf{p}}_n + 2h\gamma\mathbf{L}_{n+1}\ddot{\mathbf{p}}_{n+1} \\ 0 \end{bmatrix} \tag{12}$$

The modified Newmark formulas are given by Eqs. (3) and (12).

The modified Newmark formulas are applied to the two test cases from above. For the unconstraint rigid body which rotates with a constant angular velocity vector about one axis, no energy dissipation occurs any more, not even for the case when setting $\alpha < 0$. For the second test case, where a torque about one axis is applied to an unconstraint body being at rest, the angular velocity after one time step is given by

$$\omega_{n+1} = \frac{m_x}{I_x}h - \frac{\alpha(1-2\alpha)m_x^3}{64I_x^3}h^5 + O[h]^6 \tag{13}$$

For this test case notable improvements are achieved. Setting $\alpha = 0$, the modified Newmark supplies the physically correct result. But also when setting $\alpha < 0$, a significant reduction in the numerical damping is achieved compared to Eq. (7).

References

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