Analysis on the stability of a bicycle moving on a surface of revolution

Jiaming Xiong[1], Nannan Wang[1] and Caishan Liu[1]

[1]*State Key Laboratory for Turbulence and Complex Systems, Peking University,*
{1601111654,wangnannan,liucs}@pku.edu.cn

Bicycle is a typical multibody system with nonholonomic constraints. Many scholars have investigated bicycle dynamics and made a series of important achievements since the end of the 19th century. The stabilities of the straight motions and circular motions of an uncontrolled bicycle moving on a flat level road have been studied widely in history [1, 2, 3]. However, the existing studies focus on a special case where the ground was limited to a horizontal plane. In this paper, we study the dynamics of a bicycle moving on a surface of revolution, and analyze the stabilities of its circular motions.

Our analysis was based on the benchmark Whipple bicycle model (see Fig.1), which is a multi-rigid-body system consisting of four rigid bodies: a rear wheel, a rear frame with the rider body rigidly attached to it, a front wheel and a front frame consisting of the fork and the handlebar. Nine generalized coordinates are needed to describe the configuration of an unconstrained bicycle.

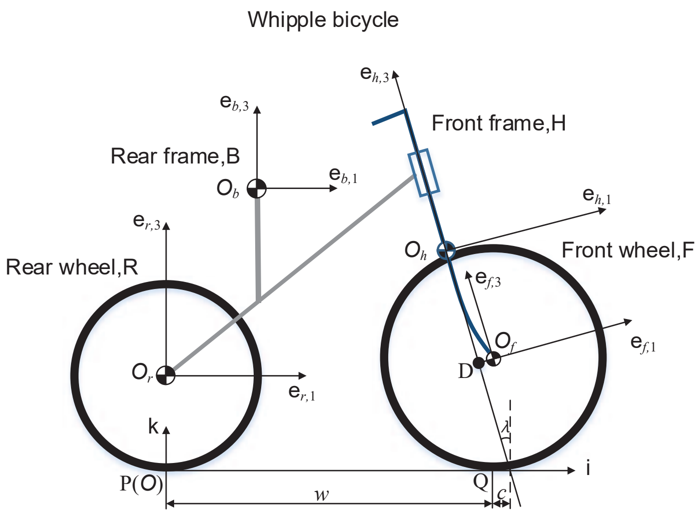

Fig. 1: The benchmark Whipple bicycle in a upright, straight reference configuration (a figure adapted from [4]).

We assume that the ground can be mathematically described as $z = f(x^2 + y^2)$ in an inertial coordinate frame, corresponding to a surface of revolution. The constraints induced by the contact interactions are closely related to the surface shapes and the motions of two contacting bodies. To study the evolutions of the contact points with respect to time, we introduce two curve parameters identifying the points on the edge of the rear and the front wheel, and four surface parameters identifying the points on the ground. Under the condition that no slippage exists between the wheels and the ground, two holonomic constraint equations as well as four nonholonomic constraint equations can be derived by using the methods proposed by Zhao and Liu in [5]. Compared to the case of a bicycle moving on a horizontal plane, the constraint equations take more complex formulations.

We adopt the Lagrangian equations to derive the governing equations of the benchmark bicycle moving on the surface of revolution. Designate by $\mathbf{q}$ as the set of the generalized coordinates, by $\mathbf{p}$ as the set of the curve and surface parameters, and by Λ as the set of the Lagrange multipliers. The governing equations of motion for the benchmark bicycle read,

$$\begin{cases} \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} = \mathbf{F}(\dot{\mathbf{q}}, \mathbf{q}) + \mathbf{V}^T(\mathbf{q}, \mathbf{p})\Lambda, \\ \mathbf{V}(\mathbf{q}, \mathbf{p})\dot{\mathbf{q}} = \mathbf{0}, \\ \mathbf{G}(\mathbf{q}, \mathbf{p}) = \mathbf{0}, \end{cases} \tag{1}$$

where $\mathbf{M} \in \mathbb{R}^{9\times 9}, \mathbf{F} \in \mathbb{R}^9, \mathbf{V} \in \mathbb{R}^{6\times 9}, \mathbf{G} \in \mathbb{R}^6$. The second equation in Eq.(1) corresponds to the holonomic and nonholonomic constraint equations subjected to the two contact points, and the third equation is related to the parameter equations that establish the relation between the geometric parameters of the contact surfaces (or curves) and the generalized coordinates of the system.

In order to study the circular motions of an uncontrolled bicycle, we introduce a series of rotation transformations to eliminate the time-dependent variables included in the set of the generalized coordinates, the surface parameters and the Lagrange multipliers. The circular solutions of the bicycle then correspond to the equilibrium points of a new DAE system. We prove theoretically that these equilibrium points are not isolated, but in the form of one-parameter families of solutions[3, 4]. Accordingly, we derive the linearized equations around the equilibrium point under small perturbations. By eliminating the dependent variables specified in constraint equations, we obtain seven first-order linear ordinary differential equations. Finally, the stability of the equilibrium point can be analyzed by calculating the eigenvalues of the linear system.

References

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