# A rigid body formulation with non-redundant unified local velocity coordinates

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In this paper, we present a formulation for spatial rigid body motion based on a set of non-redundant, homogeneous local velocity coordinates. The initial motivation for the present work was the investigation of local velocity coordinates of a first order H(curl)-conforming finite element proposed by Nédélec [1]. Since the latter finite element offers six non-redundant – regarding their physical interpretation – equivalent degrees of freedom, the results of this research turned out to be relevant for rigid bodies as well, and are presented herein.

In contrast to the common practice, the proposed approach renounces the distinction between translational and angular velocity in the sense that it only makes use of translational local velocities. To obtain the new set of coordinates, we use the velocity vectors of six properly selected points of a rigid body, and represent the rigid bodys translational and rotational velocities in terms of six local translational velocities. The points at which the six local velocities are determined are, for example, located on the six edges of a tetrahedron or on the six lateral surfaces of a hexahedron, see Fig. 1. While the linear relation to the conventional local translational and angular velocity vectors might be obvious in the hexahedral case, it is not apparent in the tetrahedral case. To obtain position coordinates and the orientation of the body, we use the exponential map on the special Euclidean group SE(3). In order to solve the incremental motion vector differential equation, we introduce the inverse of the tangent operator corresponding to our unified motion approach. Furthermore, we use a recently presented 4th-order Runge-Kutta time integration scheme [2], which we have extended in a way, such that it can be used with respect to our approach and elements of SE(3). We use a benchmark problem as a numerical example to show the applicability of our approach.

In the first part of the paper, we present the theoretical fundamentals of our approach and discuss its differences with respect to well established formulations like the natural coordinates [3] or Lie-group methods [4]. Our proposed approach unifies the description of rigid body motion on velocity level. This unification is obtained, by using the velocity vectors of six properly selected points of a rigid body, for example, located on the six edges of a tetrahedron or on the six lateral surfaces of a hexahedron, see Fig. 1. A condition for the selection of such points will be presented in the final paper. Using this approach, we obtain six fully equivalent (=unified) local velocity coordinates $\bar{\mathbf{w}} = \left[\bar{\mathrm{w}}_1\ \bar{\mathrm{w}}_2\ \bar{\mathrm{w}}_3\ \bar{\mathrm{w}}_4\ \bar{\mathrm{w}}_5\ \bar{\mathrm{w}}_6\right]^{\mathrm{T}}$.

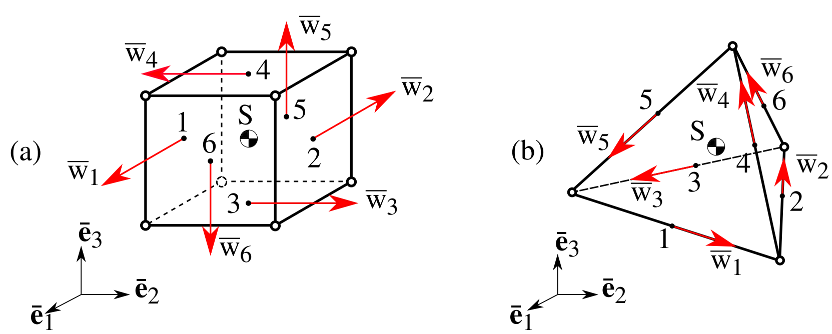


Fig. 1: Basic geometries for unified local velocity coordinates. S denotes the center of mass of the rigid body. The six points, used for the velocity projection are denoted with 1...6. The corresponding six unified velocities are denoted by $\bar{\mathrm{w}}_1 \ldots \bar{\mathrm{w}}_6$. Fig. a) shows the hexahedral case, Fig b) shows the tetrahedral case.

The second part of the paper is dedicated to the derivation of the governing equations of motion (EOM). We use the Gibbs-Appell equations [5] to obtain them, reading

$$\overline{\mathbf{M}}\,\dot{\overline{\mathbf{w}}} + \overline{\Gamma}(\overline{\mathbf{w}})\overline{\mathbf{w}} = \overline{\mathbf{Q}}(\mathbf{H}, \overline{\mathbf{w}}, t). \tag{1}$$

Herein, the mass matrix $\overline{\mathbf{M}}$ is constant. The vector $\dot{\overline{\mathbf{w}}}$ denotes the time derivative of the local velocity coordinates. The vector $\overline{\mathbf{Q}}$ summarizes all external forces and torques acting on a rigid body with respect to the local (body fixed) coordinate system. Velocity dependent loads are given in terms of the six velocity coordinates $\overline{\mathrm{w}}$. Loads, which are depending on the current position or orientation of the body, are formulated in terms of group elements $\mathbf{H} \in SE(3)$. Special time dependencies can be considered within the above mentioned framework. The vector $\overline{\Gamma}\overline{\mathbf{w}}$ represents the quadratic velocity term. Furthermore, we show how the obtained EOM can be converted into the Newton-Euler equations. In the third part of the paper, we show how our approach can be used to obtain elements of SE(3). Therefore, we introduce the inverse of the tangent operator $\mathbf{T}^{-1}_{SE(3)}$ corresponding to our unified motion approach,

$$\overline{\mathbf{T}}^{-1}_{SE(3)}(\mathbf{n}) = \mathbf{D} + \frac{1}{2}\mathbf{T}_1(\mathbf{n}) + \mathbf{K}(\mathbf{n})\mathbf{T}_2(\mathbf{n}). \tag{2}$$

The structure of the matrices $\mathbf{D}$, $\mathbf{T}_1$ and $\mathbf{T}_2$ in Eq. (2) depends on the chosen velocity coordinates. The matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ depend on the incremental motion vector $\mathbf{n}$, while $\mathbf{K}(\mathbf{n})$ is a coefficient matrix. The update of a SE(3) group element is determined by using the exponential map on SE(3),

$$\begin{aligned} \dot{\mathbf{n}} &= \overline{\mathbf{T}}^{-1}_{SE(3)}(\mathbf{n})\,\overline{\mathbf{w}} \\ \mathbf{H} &= \mathbf{H}_0 \exp_{SE(3)}(\mathbf{n}). \end{aligned} \tag{3}$$

In order to solve the ordinary differential equation of the incremental motion vector $\mathbf{n}$, we use a up-to-date time stepping scheme [2], which we adapted to solve Eq. (1) and Eq. (3). Subsequently, we demonstrate the applicability of our approach with a numerical example. Moreover, we provide convergence studies of the extended integration scheme. Finally, we discuss the results obtained from the numerical example, the advantages and disadvantages of our approach.

Concluding, we found a formulation to describe spatial rigid body motion in terms of non-redundant, homogeneous local velocity coordinates. We obtain equations of motion with a simple structure, which can be integrated using adopted Lie-group time integration schemes. Currently, the proposed approach offers no computational advantages compared to state of the art formulations but it could be useful in cases, in which one does not want to distinguish between translational and rotational motion.

## References


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