An enhanced LCP model of planer multiple contact problems with static friction

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Contact problems can be found in many mechanical systems [1]. In the modeling of mechanical systems, contacts problems may need the proper treatment due to their non-smooth nature. Some approaches have been used to deal with contact problems. And the complementarity method [2] is a quite popular one.

Applying the complementarity method, the imposed constraints of the contact problems are usually decomposed into complementarity conditions. Such complementarity conditions are incorporated into equations of motion, which are formulated as complementarity problems (CPs) [3]. In planner contact problems under Coulomb's friction law and spatial contact problems under polyhedron friction law, the governing equations can be formulated as linear complementarity problems (LCPs). Meanwhile, for spatial contact problems under cone friction law, the governing equations can be formulated as cone complementarity problems (CCPs), which belong to nonlinear complementarity problems (NCPs). The complementarity method has been studieded for several years. Whereas, models of the complementarity method may not be "perfect" enough. Some non-perfectence of the previous complementarity models is shown in the follwing example.

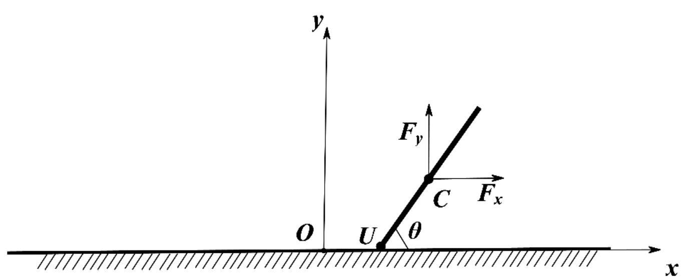

Fig. 1: A single pendulum subject to unilateral kinematic constraint

Let us consider a planner single pendulum subject to unilateral kinetic constraint at one end, as shown in Fig.1. The pendulum is uniform with total mass m and total length $2l$. At the contact point U, the initial values of normal distance, normal velocity and tangential velocity are all zero. Two external forces F_x and F_y are exerted at the center of mass C.

Applying Newton's motion law and Coulomb's friction law, equations of motion can be written as:

$$\begin{bmatrix} F_N^u \\ F_T^u \end{bmatrix} = \frac{m}{4}\begin{bmatrix} 1+3\sin^2\theta & 3\sin\theta\cos\theta \\ 3\sin\theta\cos\theta & 1+3\cos^2\theta \end{bmatrix}\begin{bmatrix} \ddot{g}_N^u \\ \ddot{g}_T^u \end{bmatrix} - \frac{1}{4}\begin{bmatrix} \left(1+3\sin^2\theta\right)F_y + 3\sin\theta\cos\theta F_x + 4ml\sin\theta\dot{\theta}^2 \\ \left(1+3\cos^2\theta\right)F_x + 3\sin\theta\cos\theta F_y + 4ml\cos\theta\dot{\theta}^2 \end{bmatrix} \tag{1}$$

$$\begin{aligned} &\text{open contact:} \quad F_N^U = 0 \quad \ddot{g}_N^U > 0; \\ &\text{closed contact:} \quad F_N^U \geq 0 \quad \ddot{g}_N^U = 0 \end{aligned} \tag{2}$$

$$\begin{aligned} &\text{sliding:} \quad F_T^U = -\operatorname{sgn}\left(\ddot{g}_T^U\right)\mu F_T^U \qquad \left|\ddot{g}_T^U\right| > 0; \\ &\text{sticking:} \quad \left|F_T^U\right| \leq \mu F_N^U \qquad \left|\ddot{g}_T^U\right| = 0 \end{aligned} \tag{3}$$

It can be verified that Eq.(2) is equivalent to a complementarity condition:

$$\ddot{g}_N^U \geq 0, \quad F_N^U \geq 0, \quad F_N^U \ddot{g}_N^U = 0 \tag{4}$$

As for Eq.(3), the terms F_{Ti}^{u} and $\ddot{g}_{Ti}^{u}$ are decomposed as:

$$\ddot{g}_T^{U+} = \left(\left|\ddot{g}_T^U\right| + \ddot{g}_T^U\right)/2 \quad \ddot{g}_{Ti}^{U-} = \left(\left|\ddot{g}_T^U\right| - \ddot{g}_T^U\right)/2; \quad F_T^{U+} = \left(\mu F_N^U + F_T^U\right)/2 \quad F_{Ti}^{U-} = \left(\mu F_N^U - F_T^U\right)/2 \tag{5}$$

Then, two complementarity conditions are derived from Eq.(3) and Eq.(5):

$$\ddot{g}_T^{U+} \geq 0, \quad F_T^{U+} \geq 0, \quad F_T^{U+}\ddot{g}_T^{U+} = 0; \qquad \ddot{g}_T^{U-} \geq 0, \quad F_T^{U-} \geq 0, \quad F_T^{U-}\ddot{g}_T^{U-} = 0 \tag{6}$$

In the previuos complementarity models, the complementarity conditions Eq.(4) and Eq.(6) are added into Eq.(1) and they can be transformed as a LCP:

$$\begin{aligned} &\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{p} \quad \boldsymbol{y} \geq 0 \quad \boldsymbol{x} \geq 0 \quad \boldsymbol{y}^{\mathrm{T}}\boldsymbol{x} = 0 \\ &\boldsymbol{x} = \begin{bmatrix} \ddot{g}_N^U & \ddot{g}_T^{U+} & \ddot{g}_T^{U-} \end{bmatrix}^{\mathrm{T}}; \quad \boldsymbol{y} = \begin{bmatrix} F_N^U & F_T^{U+} & F_T^{U-} \end{bmatrix}^{\mathrm{T}} \end{aligned} \tag{7}$$

In Eq.(7), since the terms $\ddot{g}_T^{U+}$ and $\ddot{g}_T^{U+}$ are defined as $\ddot{g}_T^{U+} = \left(\left|\ddot{g}_T^U\right| + \ddot{g}_T^U\right)/2; \quad \ddot{g}_{Ti}^{U-} = \left(\left|\ddot{g}_T^U\right| - \ddot{g}_T^U\right)/2$, the condition $\ddot{g}_T^{U+}\ddot{g}_{Ti}^{U-} = 0$ should be obeyed all the time. However, $\ddot{g}_T^{U+}\ddot{g}_{Ti}^{U-} = 0$ has not been added in Eq.(7). The missing of $\ddot{g}_T^{U+}\ddot{g}_{Ti}^{U-} = 0$ may lead that Eq.(7) has some "unnecessary" solutions.

Let us set the initial conditions as $\theta = \pi/4, \dot{\theta} = 0, \mu^u = 1, \dot{x} = 0, \dot{y} = 0, F_x = 0, F_y > 0$. Then we may find that Eq.(1)-Eq.(3) has a unique solution: $F_N^U = 0, F_T^U = 0, \ddot{g}_T^U = 0, \ddot{g}_N^U = F_y/m$. While for Eq.(7), it can be verified that there is a set of solutions $\boldsymbol{x} = \begin{bmatrix} F_y/m & \alpha & \alpha \end{bmatrix}^{\mathrm{T}}; \boldsymbol{y} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, $\alpha \geq 0$ which contain an infinite number of "unnecessary" solutions with any $\alpha > 0$. The "unnecessary" solutions may cause waste of computation as well as other numerical problems. If we add $\ddot{g}_T^{U+}\ddot{g}_{Ti}^{U-} = 0$ into Eq.(7), the new LCP would have an unique solution: $\boldsymbol{x} = \begin{bmatrix} F_y/m & 0 & 0 \end{bmatrix}^{\mathrm{T}}; \quad \boldsymbol{y} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$ which is in one-to-one correspondence to the unique solution of Eq.(1)-Eq.(3). Such phenomenon implies that $\ddot{g}_T^{U+}\ddot{g}_{Ti}^{U-} = 0$ is indeed a necessary condition which should be added into the complementarity model. The above discusssion is about a single contact problem. In fact, such discussion can be expanded into multiple contact problems with both unilateral and bilateral kinetic constraints. And the conclusions are similar.

In this paper, an enhanced LCP model of planner multiple contact problems which adds the missed conditions is proposed. The proposed model is proved to be completely equivalent to the concerned contact problems (without any "unnecessary" solutions which appear in the above example). Besides, complementarity theory is applied to investigate properties of solutions of the proposed model. Existence of solutions and boundedness of solutions are proven, so the presented model are always solvable. Sufficient conditions of uniqueness and finiteness of number of solutions are provided. These conditions are applied in parametric study. Finally, several numerical examples are given to show non-uniqueness or infiniteness of number of solutions, which may be related to some non-smooth phenomena [4].

References

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