Rigid Bodies in Continuum Mechanics

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The actual author has proposed to treat rigid bodies as continua whatever the nature of parameters, e.g. quaternions q a-priori incompatible with rigidity. But it is necessary to eliminate the stress tensor, so introduced in the virtual work principle. Here we propose to show the applicability of our method to an example involving friction expressed by inequality relations.

1. **Foundations.** If motions of rigid bodies are described by parameters (like quaternions here chosen) incompatible with rigidity, rigidity constraints must be explicit [1]. Here we propose to extend Lagrange method when rigidity law is impressed by explicit relations, recalling that usual Lagrange equations [2] (eventually with multipliers) are founded under the independence of parameters.

Choice of Displacements and Virtual Velocity Fields. Being concerned by one rigid body B , we introduce available motions (H)

$$x=R(q(t))X \quad , \quad \dot{x}=(R'_i\, R^{-1}x)\dot{q}_i \quad , \quad v=(R'_i\, R^{-1}x)w_i \qquad \text{(H)}$$

(sum on repeated indices) where R is some invertible 3×3-matrix, function of n independent parameters q_i , R'_i is the partial derivative of $R(q_1,\ldots,q_n)$ and x the actual position of the particle X. The w_i's are arbitrary virtual velocities. R is not necessarily a rotation: e.g. since quaternions are used, then the rigidity constraint $q^Tq=1$ is not fulfilled.

Virtual Work Principle (VWP). Since rigidity is not satisfied, it results the existence of strains. So the VWP available in Continuum Mechanics is required [3] –

$$\int_B \rho\, a.v\, dx + \int_\Gamma f.v\, da - \int_B \sigma : grad\ v\, dx = 0$$

whatever v (σ : Cauchy stress tensor, f: surface forces on Γ (no volume forces for simplicity)). The first term is the virtual work (denoted L_iw_i) of acceleration a, where L_i is obtained by Lagrange usual formula as a function of kinetic energy. We have

$$grad\ v= (R'_i\, R^{-1})w_i=S_iw_i+A_iw_i \quad , \quad \sigma{:}grad v =(\sigma{:}S_i\,)w_i$$

where S_i and A_i are resp. the symmetric and anti-symmetric parts of the matrix $R'_i\, R^{-1}$. The last equality does not contain the matrix A_i since σ is symmetric and A_i is anti-symmetric.

Elimination of stress tensor. Now in order to eliminate the Cauchy stress tensor, we require the relations $S_iw_i=0$ (sum on i). In addition, it is seen that surface forces f occur by global quantities only (i.e. $R(f)$ and $M(f)$). Note that this elimination is a-priori realised if R is a rotation since $RR^T=Id$ implies that $S_i=0$ for $i=1,\ldots,n$. So the following compatibility conditions result: whatever the w_i's such that $S_iw_i=0$, we have

$$[-L_i+M(f)\, a_i]w_i=0 \quad \text{(sum on } i\text{)}$$

(a_i: dual vector of A_i) under the only above hypothesis (H).

Material constitutive law. Finally we write the rigidity constraint (the material constitutive law) $q^Tq=1$ if quaternions. It is noteworthy that no undue hypothesis on the virtual work of internal forces were made in our paper.

2. **Example: contact with friction.** We consider an homogeneous rigid wheel (centre O, radius r and mass m) rolling in a vertical plane $O_0x_0y_0$ on an inclined line (or surface) O_0X_0 under the gravitational acceleration g downwards, the gravitational force being $(f=-mgy_0)$ applied on the centre O of the wheel. We

use the referential $Ref=O_0X_0Y_0Z_0$ with the angle between O_0x_0 and O_0X_0 noted a . Two-dimensional Euler parameters (p,q) are introduced to specify the rotation of the wheel, so writing for the matrix R

$$R_{11}=R_{22}=1-2q^2 \ , \ R_{12}=-R_{21}=-2pq \ , \ R^{-1}=R^T/\Delta \quad , \quad \Delta =1+4q^2(p^2+q^2-1$$

Now we introduce the virtual coefficients (w_x,w_y,w_p,w_q) associate to the parameters (x,y,p,q) and the condition $w_iS_i=0$, i.e. $pw_p+qw_q=0$. Under the above condition, the VWP is writing

$$-\int_B \rho a.v\,dx - mgy_0 .v(O)+Tv_1(A)+Nv_2(A)=0$$

where $(T,N,0)$ are the components of the two-dimensional contact force on the wheel applied at the contact point A. Now we must use the contact law of friction, by example in the hypothesis of a bilateral contact $(y=r)$ at the point $A=(x,y-r,0)$ of the wheel, implying the geometric constraint $y=r$, together with the Coulomb law of friction equivalent to the inequality [4]

$$T\,[v_1(A)-u_1(A)]+k\,|N|\,[|v_1(A)|-|u_1(A)|]\geq 0$$

First the parameters are specified such that $w_x=w_p=w_q=0$, satisfying (22) (i.e. $w_iS_i=0$). It results $v(x)=(0,w_y,0)$ so that by taking account of the bilateral contact $y=r$

$$(mgcosa-N)=0 \text{ and } \dot{K} +mgsina\ \dot{x} + k\,|N|\,|u_1(A)| =0$$

$$\int_B \rho a.v\,dx +mgy_0\, v(O)-Nv_2(A)+ k\,|N|\,|v_1(A)| \geq 0 \quad \text{where } N=mg\cos a$$

that is available whatever the parameters (w_x,w_p,w_q). After some straightforward calculus, the acceleration term is obtained under the form

$$\int_B \rho a.v\,dx = m\ddot{x}w_x+a_{11}\ddot{p}+a_{22}\ddot{q}+2a_{12}\dot{p}\dot{q}+2b\dot{q}^2$$

$$a_{11}=2mr^2(q^2w_p+pqw_q),\ a_{22}=2mr^2(pq+p^2+4q^2)w_q,\ a_{12}=2mr^2(qw_p+pw_q),\ b=4mr^2qw_q$$

Taking account of this expression, the differential variational inequality follows

$$(m\ddot{x}+mg\sin a)w_x+kmg\cos a\,|w_x+r(\alpha_p w_p+\alpha_q w_q)|+2mr^2(Aw_p+Bw_q)\geq 0$$

(where α_p,α_q and A,B are given functions) under the compatibility condition $pw_p+qw_q=0$. That is the basic relation to solve the problem completed naturally by initial conditions on velocities (and positions). The numerical treatment of this inequality is not the aim of this present mechanical work [5].

3. Conclusion and References. The present paper has presented a natural link existing between Analytical Dynamics and Continuum Mechanics. The keys of the actual scheme were the use of the Virtual Work Principal; then the elimination of Cauchy stresses introduces compatibility relations between virtual coefficients. The example has highlighted the necessity to separate these compatibility conditions and the relations expressing mechanical laws

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