The motions of the celt on a horizontal plane with viscous friction

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The motion of the celt attracts researchers attention by unusual properties. If we put the celt on a horizontal plane and twist it about its vertical axis, then, after a certain time, it stops rotating about this axis, but vibrations about the other axes may arise and then the celt begins rotating about the vertical axis in the opposite direction. The known studies of the celt motion contain some simplifying assumptions. For example, it is usually assumed that the speed of a contact point with plane is zero (non-holonomic formulation of the problem). In the present paper, it is assumed that at the point of contact of the celt with the plane, in addition to the normal reaction, there is also a force proportional to the velocity of this point of the body and acting in the opposite direction (the viscous friction force).

We will model the celt with an nonhomogeneous ellipsoid of rotation. It is assumed that the mass centre of the ellipsoid is on its axis of symmetry and the equation of the spheroid surface in its principal central axes $C\xi_1\xi_2\xi_3$ has the form

$$f(\mathbf{r}) = \frac{(\xi_1 \sin\varphi + \xi_2 \cos\varphi)^2}{a^2} + \frac{(\xi_1 \cos\varphi - \xi_2 \sin\varphi)^2}{b^2} + \frac{(\xi_3 - d)^2}{c^2} = 0.$$

In a moving system of coordinates $C\xi_1\xi_2\xi_3$ the equations of motion of the body have the form

$$m\dot{\mathbf{v}} + [\omega, m\mathbf{v}] = -mg\gamma + N\gamma + \mathbf{F} \tag{1}$$

$$\mathbf{J}\dot{\omega} + [\omega, \mathbf{J}\omega] = [\mathbf{r}, N\gamma + \mathbf{F}] \tag{2}$$

$$\dot{\gamma} + [\omega, \gamma] = 0 \tag{3}$$

$$(\mathbf{u}, \gamma) = 0 \tag{4}$$

and express theorems on the change in the momentum (1) and in the angular momentum (2), the constancy of the unit vector of the ascending vertical γ (3)

$$\gamma = -\frac{\operatorname{grad} f(\mathbf{r})}{|\operatorname{grad} f(\mathbf{r})|}$$

and the condition of non-detachment of the body motion (4). Here m is the mass of the spheroid, $\mathbf{v}$ is the velocity vector of its centre of the mass of the spheroid, ω is the angular velocity vector, g is the gravitational acceleration, N is the value of the normal component of the reaction of the supporting plane, $\mathbf{J} = \operatorname{diag}(\mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_3)$ is the central inertia tensor of the spheroid, $\mathbf{r}$ is the radius vector of the velocity of the point of contact of the spheroid with the plane,$\mathbf{u} = \mathbf{v} + [\omega, \mathbf{r}]$ is the velocity of this point, $\mathbf{F} = -m\kappa\mathbf{u}$ is the viscous friction force and κ is the friction coefficient.

Note that the value $\kappa = 0$ corresponds to the case of an absolutely smooth surface. Moreover, in the limit case when $\kappa = \rightarrow \infty$ the viscous friction force produces non-holonomic formulation of the problem, corresponding to the case of an absolutely rough surface [1].

The system (1)–(4) has the solution

$$\mathbf{v} = 0, \quad \gamma_1 = \gamma_2 = 0, \quad \gamma_3 = 1, \quad \omega_1 = \omega_2 = 0, \quad \omega_3 = \omega = \text{const}. \tag{5}$$

It is corresponded to uniform rotations of the celt around the axis of dynamic symmetry $C\xi_3$, coinciding with the vertical. The stability of these rotations depends from the direction of the angular velocity of rotation. For example, on the Fig. 1 the numerical solutions of the system (1)–(4) (only for the dependence of the component of angular velocity ω_3 from the time) are present for a celt with the following parameters

$$m = 1 \text{ kg}, a = 0.09 \text{ m}, b = c = 0.03 \text{ m}, d = 0.015 \text{ m}, \varphi = 0.15$$

$$A_1 = 0.00075\ kg \cdot m^2, A_2 = 0.00456\ kg \cdot m^2, A_3 = 0.00485\ kg \cdot m^2,$$

for the following initial conditions

$$\mathbf{v}(0) = 0, \quad \gamma_2(0) = 0, \quad \gamma_3(0) = 0.99, \quad \omega_1(0) = \omega_2(0) = 0, \quad \omega_3(0) = \pm 5\ s^{-1}.$$

Thus, the rotations are stable if the angular velocity is co-directed with the vertical, and the rotations are unstable if the celt is rotated in the opposite direction. In latter case, the direction of rotation is reversed.

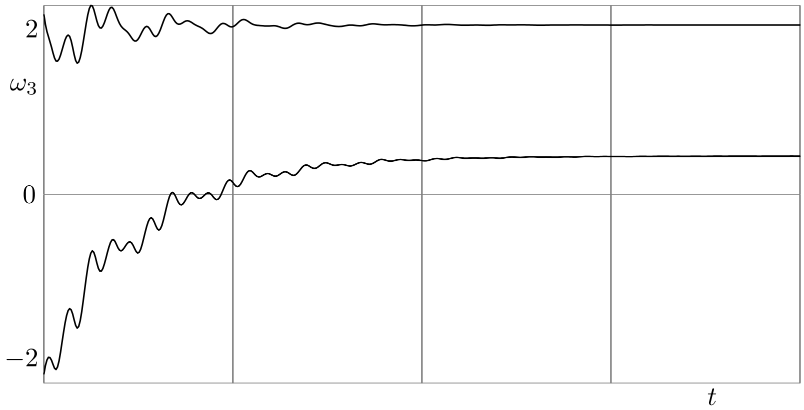

Fig. 1: Numerical experiments

It is well known [1], [2], [3] that in the case of a nonholonomic formulation of the rotation problem for changing the direction of rotation, the parameters of the celt must satisfy the conditions

$$\begin{gathered} 0 < \varphi < \pi/2, \quad A_1 < A_2 < A_3, \quad a_1 > a_2 > a_3 \\ J = (A_1 + A_2 - A_3)\left(\frac{a_1}{a_3} + \frac{a_2}{a_3} - 2\right) - m\left(a_3^2 - 3a_3(a_1 + a_2) + 2a_1 a_2\right) > 0 \end{gathered} \tag{6}$$

(here $a_1 = a^2/c,\ a_2 = b^2/c,\ a_3 = c - d$ are the radiuses of curvature of the body surface at the point of contact). In the case of viscous friction, conditions (6) are insufficient. But if also $\varphi << 1$, $A_1 << A_3$, $a_3 >> a_1$, $a_2 >> a_1$ then the region of stability of rotations (5) has the form $\omega_1 < \omega < \omega_2$, here $\omega_1 < 0,\ |\omega_1| << 1,\ \omega_2 > 0$. Thus, if the initial angular velocity is enough big and its direction is opposite to the vertical, then the direction of rotation changes.

This research was financed by the Russian Foundation for Basic Research (16-01-00338, 18-01-00335).

References

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