Higher-order acceleration center of rigid body motion

Daniel Condurache

Department. of Theoretical Mechanics, Technical University of Iasi, daniel.condurache@tuiasi.ro

Let be the rigid motion given by the below parametic equation:

$$\begin{cases} \boldsymbol{\rho}_{\mathrm{Q}} = \boldsymbol{\rho}_{\mathrm{Q}}(t) \\ \boldsymbol{R} = \boldsymbol{R}(t) \end{cases}, t \in \underline{\mathbf{I}} \subseteq \mathbb{R} \tag{1}$$

where $\boldsymbol{\rho}_{\mathrm{Q}} = \boldsymbol{\rho}_{\mathrm{Q}}(t) \in \mathbf{V}_3$ and $\mathbf{R} = \mathbf{R}(t) \in S\mathbb{O}_3$. The spatial twist of the rigid body is defined by the pair of vectors $(\boldsymbol{\omega}, \mathbf{v})$, where:

$$\begin{gathered} \boldsymbol{\omega} = \mathrm{vect}\dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}} \\ \mathbf{v} = \dot{\boldsymbol{\rho}}_{\mathrm{Q}} - \boldsymbol{\omega} \times \boldsymbol{\rho}_{\mathrm{Q}} \end{gathered} \tag{2}$$

The n-th order acceleration of that point of the rigid body, located by the position vector $\boldsymbol{\rho}$ and denoted by $\mathbf{a}_{\boldsymbol{\rho}}^{[\boldsymbol{n}]}, n \in \mathbb{N}^*$, is

$$\mathbf{a}_{\boldsymbol{\rho}}^{[\boldsymbol{n}]} \overset{\mathrm{def}}{=} \frac{\mathrm{d}^{\boldsymbol{n}}\boldsymbol{\rho}}{\mathrm{d}t^{n}} = \boldsymbol{\rho}^{(\boldsymbol{n})} = \mathbf{a}_{Q}^{[n]} + \boldsymbol{\Phi}_{\boldsymbol{n}}\left[\boldsymbol{\rho} - \boldsymbol{\rho}_{\mathrm{Q}}\right], n \in \mathbb{N}^* \tag{3}$$

where $\mathbf{a}_{Q}^{[n]}$ is the n-order acceleration of the body-fixed point Q and $\boldsymbol{\Phi}_{\boldsymbol{n}} = \boldsymbol{R}^{(\boldsymbol{n})}\boldsymbol{R}^{T}$ represents the n-th order acceleration tensor.

The equation (3) may be written as

$$\mathbf{a}_{\boldsymbol{\rho}}^{[\boldsymbol{n}]} - \boldsymbol{\phi}_{\boldsymbol{n}}\boldsymbol{\rho} = \mathbf{a}_{Q}^{[n]} - \boldsymbol{\Phi}_{\boldsymbol{n}}\boldsymbol{\rho}_{\mathrm{Q}}, n \in \mathbb{N}^*. \tag{4}$$

This shows us that the vector function

$$\mathbf{I}_n = \mathbf{a}_{\boldsymbol{\rho}}^{[\boldsymbol{n}]} - \boldsymbol{\Phi}_{\boldsymbol{n}}\boldsymbol{\rho}, n \in \mathbb{N}^* \tag{5}$$

has the same value in every point of the rigid body under the general spatial motion, at a given moment of time *t*. It represents a **vector invariant** of the n-th order acceleration field.

The invariant value of vector $\mathbf{I}_n$ is obtained for $\boldsymbol{\rho} = \mathbf{0}$ and it is the n-th order acceleration of the point of the rigid body that passes the origin of the fixed reference frame at a given moment of time: $\mathbf{I}_n = \mathbf{a}_0^{[\boldsymbol{n}]} \overset{\mathrm{def}}{=} \mathbf{a}_{\mathrm{n}}$. The Eq. (5) becomes:

$$\mathbf{a}_{\boldsymbol{\rho}}^{[\boldsymbol{n}]} = \mathbf{a}_{\mathrm{n}} + \boldsymbol{\phi}_{\boldsymbol{n}}\boldsymbol{\rho}. \tag{6}$$

Let $\boldsymbol{\Phi}_{\boldsymbol{n}}^*$ be the adjugate tensor of $\boldsymbol{\Phi}_{\boldsymbol{n}}$ uniquely defined by:$\boldsymbol{\Phi}_{\boldsymbol{n}}\boldsymbol{\Phi}_{\boldsymbol{n}}^* = (\det\boldsymbol{\Phi}_{\boldsymbol{n}})\boldsymbol{I}$.

From Eq. (5), results another invariant

$$\boldsymbol{J}_n = \boldsymbol{\Phi}_{\boldsymbol{n}}^*\mathbf{a}_{\boldsymbol{\rho}}^{[\boldsymbol{n}]} - (\det\boldsymbol{\Phi}_{\boldsymbol{n}})\boldsymbol{\rho}, n \in \mathbb{N}^*. \tag{7}$$

The value of this invariant is $\boldsymbol{J}_n = \boldsymbol{\Phi}_{\boldsymbol{n}}^*\mathbf{a}_{\mathrm{n}}$.

In the specific case when tensor $\boldsymbol{\Phi}_{\boldsymbol{n}}$ is non-singular ($\det\boldsymbol{\Phi}_{\boldsymbol{n}} \neq 0$), from (6) results the position vector having an imposed n-th order acceleration $\mathbf{a}^*$:

$$\boldsymbol{\rho}^* = \boldsymbol{\Phi}_n^{-1}(\mathbf{a}^* - \mathbf{a}_{\mathrm{n}}), n \in \mathbb{N}^*. \tag{8}$$

In a particular case of the **n-th order acceleration centre** G_{n} (i.e. the point that have $\mathbf{a}^* = \mathbf{0}$) on obtain:

$$\boldsymbol{\rho}_{G_n} = -\boldsymbol{\Phi}_n^{-1}\mathbf{a}_{\mathrm{n}} \tag{9}$$

Assuming that the tensor $\boldsymbol{\Phi_n}$ is non-singular, the previous relations lead to a new vector invariant that characterise the accelerations of n-th and m-th order ($n, m \in \mathbb{N}^*$):

$$\boldsymbol{K_{m,n}} = \mathbf{a}_{\rho}^{[m]} - \boldsymbol{\Phi_m}\boldsymbol{\Phi}_n^{-1}\mathbf{a}_{\rho}^{[n]}, m, n \in \mathbb{N}^*. \tag{10}$$

The value of this invariant is $\boldsymbol{K_{m,n}} = \mathbf{a}_{\mathrm{m}} - \boldsymbol{\Phi_m}\boldsymbol{\Phi}_n^{-1}\mathbf{a}_{\mathrm{n}}$.

The problem of the determination the adjugate tensor of the n-th acceleration tensor and the conditions in which these tensors are inversable is, as the autor knows, still an open problem in theoretical kinematics field. We will propose a method based on the tensors algebra that will give a closed form, free of coordinate solution, dependendent to the time derivative of spatial twist.

Let $\boldsymbol{\Phi} \in \mathbf{L}(\boldsymbol{V_3}, \boldsymbol{V_3})$ a tensor and we note $\mathbf{t} = \mathrm{vect}\boldsymbol{\Phi}$ and $\mathbf{S} = \mathrm{sym}\boldsymbol{\Phi}$. The below theorem takes place.

Theorem 1. *The adjugate tensor and determinant of the tensor* $\boldsymbol{\Phi}$ *is*:

$$\begin{gathered} \boldsymbol{\Phi}^* = \mathbf{S}^* - \widetilde{\mathbf{St}} + \mathbf{t} \otimes \mathbf{t} \\ \det\boldsymbol{\Phi} = \det\mathbf{S} + \mathbf{tSt} \end{gathered} \tag{11}$$

Let $\boldsymbol{\Phi}_n$ the n-th order acceleration tensor, $\boldsymbol{\Phi}_n = \tilde{\mathbf{t}}_n + \mathbf{S}_n$.

The vectors $\mathbf{t}_n$ and the symmetric tensors $\mathbf{S}_n, n \in \mathbb{N}^*$ can be obtained with the below recurrence relation:

$$\begin{cases} \mathbf{t}_{n+1} = \dot{\mathbf{t}}_n + \frac{1}{2}[(\mathrm{trace}\boldsymbol{\Phi}_n)\mathbf{I} - \boldsymbol{\Phi}_{\boldsymbol{n}}^{\mathrm{T}}]\boldsymbol{\omega} \\ \mathbf{t}_1 = \boldsymbol{\omega} \end{cases} \tag{12}$$

$$\begin{cases} \mathbf{S}_{n+1} = \dot{\mathbf{S}}_n + \mathrm{sym}(\boldsymbol{\Phi}_n\widetilde{\boldsymbol{\omega}}) \\ \mathbf{S_1} = \mathbf{0} \end{cases} \tag{13}$$

It follows that:

- Velocity field: $\boldsymbol{\Phi}_1 = \widetilde{\boldsymbol{\omega}}, \mathbf{t}_1 = \boldsymbol{\omega}$, $\mathbf{S_1} = \mathbf{0}$

$$\begin{gathered} \boldsymbol{\Phi}_1^* = \boldsymbol{\omega} \otimes \boldsymbol{\omega} \\ \det\boldsymbol{\Phi}_1 = 0 \end{gathered} \tag{14}$$

- Acceleration field: $\boldsymbol{\Phi}_2 = \widetilde{\boldsymbol{\omega}}^2 + \dot{\widetilde{\boldsymbol{\omega}}}, \mathbf{t}_2 = \dot{\boldsymbol{\omega}}, \mathbf{S}_2 = \widetilde{\boldsymbol{\omega}}^2$

$$\begin{gathered} \boldsymbol{\Phi}_2^* = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})^2 - \widetilde{\widetilde{\boldsymbol{\omega}}^2\dot{\boldsymbol{\omega}}} + \dot{\boldsymbol{\omega}} \otimes \dot{\boldsymbol{\omega}} \\ \det\boldsymbol{\Phi}_2 = -(\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}})^2 \end{gathered} \tag{15}$$

- Jerk field: $\boldsymbol{\Phi}_3 = \ddot{\widetilde{\boldsymbol{\omega}}} + 2\dot{\widetilde{\boldsymbol{\omega}}}\widetilde{\boldsymbol{\omega}} + \widetilde{\boldsymbol{\omega}}\dot{\widetilde{\boldsymbol{\omega}}} + \widetilde{\boldsymbol{\omega}}^3, \mathbf{t}_3 = \ddot{\boldsymbol{\omega}} + \frac{1}{2}\dot{\boldsymbol{\omega}} \times \boldsymbol{\omega} - \boldsymbol{\omega}^2\boldsymbol{\omega},\ \mathbf{S_3} = \frac{3}{2}\left[\widetilde{\boldsymbol{\omega}}\dot{\widetilde{\boldsymbol{\omega}}} + \dot{\widetilde{\boldsymbol{\omega}}}\widetilde{\boldsymbol{\omega}}\right]$,

$$\boldsymbol{\Phi}_3^* = \frac{9}{4}[(\boldsymbol{\omega} \otimes \dot{\boldsymbol{\omega}})^2 + (\dot{\boldsymbol{\omega}} \otimes \boldsymbol{\omega})^2 + (\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}}) \otimes (\dot{\boldsymbol{\omega}} \times \boldsymbol{\omega})] - \widetilde{\mathbf{St}_3} + \mathbf{t}_3 \otimes \mathbf{t}_3 \tag{16}$$

$$\det\boldsymbol{\Phi}_3 = \frac{12(\mathbf{t}_3 \times \dot{\boldsymbol{\omega}})(\boldsymbol{\omega} \times \mathbf{t}_3) + 27\boldsymbol{\omega} \cdot \dot{\boldsymbol{\omega}}(\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}})^2}{4}$$

The hyper-jerk field $\boldsymbol{\Phi}_4$ will be obtained in a similar way.

References

[1] J. Angeles, “Fundamentals of Robotic Mechanical Systems” Springer Verlag, New-York-Berlin-Heidelberg, 1997

[2] J. Angeles, “The Angular Acceleration Tensor of Rigid-Body Kinematics and its Properties”, Archive of Applied Mechanics, 69 , 204-214, 1999

[3] D. Condurache and M. Matcovschi, “Computation of Angular Velocity and Acceleration Tensors by Direct Measurements”, Acta Mechanica, vol. 153, Issue 3–4, pp 147–167, 2002

[4] D. Condurache and M. Matcovschi, “Algebraic Computation of the Twist of a Rigid Body Through Direct Measurements”, Computer Methods in Applied Mechanical Engineering , 190/40-41 , 5357-5376, 2001