Constrained systems in multibody dynamics

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Representing joints in a multibody system model by holonomic constraints

Joints are modelling elements in multibody system dynamics that restrict the relative motion of neighbouring bodies locking one or more degrees of freedom. As modelling elements, they are idealized in the sense that any clearance in real physical joints is neglected. In a simplified setting, the reaction forces in physical joints are given by

$$\mathbf{F}_\varepsilon(\mathbf{q}_\varepsilon) = -\frac{1}{\varepsilon^2}\nabla_{\mathbf{q}}\|\mathbf{g}(\mathbf{q}_\varepsilon)\|_2^2 \tag{1}$$

with $\mathbf{q}_\varepsilon(t) \in \mathbb{R}^k$ denoting the position coordinates, a small parameter $\varepsilon > 0$ and a function $\mathbf{g} : \mathbb{R}^k \to \mathbb{R}^m$, $(m \leq k)$, representing the joint geometry. Neglecting joint clearance, we have to consider the limit case $\varepsilon \to 0$ and get $\mathbf{q}_\varepsilon(t) \to \mathbf{q}(t)$, $\mathbf{F}_\varepsilon(\mathbf{q}_\varepsilon) \to \mathbf{F}_0(\mathbf{q},\boldsymbol{\lambda})$ with constraint forces $\mathbf{F}_0(\mathbf{q},\boldsymbol{\lambda}) = -\mathbf{G}^\top(\mathbf{q})\boldsymbol{\lambda}$ in the differential-algebraic model equations

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} = \mathbf{f}(t,\mathbf{q},\dot{\mathbf{q}}) - \mathbf{G}^\top(\mathbf{q})\boldsymbol{\lambda}\,, \tag{2a}$$

$$\mathbf{0} = \mathbf{g}(\mathbf{q})\,. \tag{2b}$$

Here, $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{k\times k}$ denotes the mass matrix, $\mathbf{f}(t,\mathbf{q},\dot{\mathbf{q}}) \in \mathbb{R}^k$ the force vector, $\mathbf{G}(\mathbf{q}) := (\partial\mathbf{g}/\partial\mathbf{q})(\mathbf{q}) \in \mathbb{R}^{m\times k}$ the constraint Jacobian and $\boldsymbol{\lambda}(t) \in \mathbb{R}^m$ the vector of Lagrange multipliers. For smooth constraint functions $\mathbf{g}$, the holonomic constraints (2b) imply *hidden* constraints

$$\mathbf{0} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g}\big(\mathbf{q}(t)\big) = \frac{\partial\mathbf{g}}{\partial\mathbf{q}}\big(\mathbf{q}(t)\big)\dot{\mathbf{q}}(t) = \mathbf{G}(\mathbf{q})\dot{\mathbf{q}} =: \dot{\mathbf{g}}(\mathbf{q},\dot{\mathbf{q}})\,, \tag{3a}$$

$$\mathbf{0} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{g}\big(\mathbf{q}(t)\big) = \mathbf{G}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{g}_{\mathbf{qq}}(\mathbf{q})\big(\dot{\mathbf{q}},\dot{\mathbf{q}}\big) =: \ddot{\mathbf{g}}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}}) \tag{3b}$$

at the level of velocity and acceleration coordinates, respectively.

If $\operatorname{rank}\mathbf{G}(\mathbf{q}) = m$ then constraints (2b) are non-redundant and free of contradictions and we get $\ddot{\mathbf{q}} = \ddot{\mathbf{q}}(t,\mathbf{q},\dot{\mathbf{q}})$, $\boldsymbol{\lambda} = \boldsymbol{\lambda}(t,\mathbf{q},\dot{\mathbf{q}})$ summarizing (2a) and (3b) in a block structured linear system that is uniquely solvable if the symmetric, positive semi-definite mass matrix $\mathbf{M}(\mathbf{q})$ is positive definite at $\ker\mathbf{G}(\mathbf{q})$. Therefore, the constrained system (2) is uniquely solvable for any consistent initial values $\mathbf{q}_0$, $\dot{\mathbf{q}}_0$ satisfying $\mathbf{0} = \mathbf{g}(\mathbf{q}_0) = \dot{\mathbf{g}}(\mathbf{q}_0,\dot{\mathbf{q}}_0)$. All classical results of the theory of ordinary differential equations (ODEs) apply as well to constrained systems (2) as long as the smoothness and full rank assumptions on $\mathbf{g}(\mathbf{q})$, $\mathbf{G}(\mathbf{q})$ and $\mathbf{M}(\mathbf{q})$ are satisfied. This statement remains valid if the *scleronomic* constraints (2b) are substituted by *rheonomic* constraints $\mathbf{0} = \mathbf{g}\big(t,\mathbf{q}(t)\big)$.

Time integration of constrained systems

In the classification scheme of differential-algebraic equations (DAEs), the equations of motion (2) form an index-3 DAE. Its direct time discretization requires special care to avoid numerical instability and large error terms in transient phases after initialization and step size changes [1, 2]. Therefore, it is common practice to use one or more of the hidden constraints (3) for the robust and efficient time integration of constrained systems [2, 3].

Stability and convergence of the numerical solvers may be studied by a perturbation analysis that shows the complex mechanisms of error propagation and error amplification for constrained systems. There are essential differences to ODE time integration and an inherent risk of numerical instability that affects not only monolithic solvers but also all modular methods in co-simulation frameworks etc.

Error bounds for the time integration of constrained systems (2) are furthermore the key to the convergence analysis of implicit solvers for unconstrained systems with (very) stiff potential forces (1). Linear time invariant systems of this type may be decoupled into stiff and non-stiff directions and the high-frequency oscillations in the stiff components are damped out by numerical dissipation. In the nonlinear case, the stiff and non-stiff components are coupled by higher order terms resulting in an energy transfer from high-frequency to low-frequency modes.

Constrained systems in multibody dynamics: Beyond the classical setting

Nonlinear configuration spaces From the pure mathematical viewpoint, constrained systems (2) are second order index-3 DAEs in the linear configuration space $\mathbb{R}^k$ with a special structure of the algebraic part. This type of DAEs is obtained as well for equations of motion in a lower dimensional *nonlinear configuration space* that is embedded in $\mathbb{R}^k$ with normalization or orthogonality conditions that may be written as (2b). In that case, there is *no* physical justification for accepting (small) constraint residuals in time integration. For nonlinear configuration spaces with Lie group structure, there are *Lie group integrators* that enforce such normalization or orthogonality conditions (2b) by construction. These methods are available as well for constrained systems on Lie groups [4].

Non-holonomic constraints, Servo constraints Time integration methods for constrained systems (2) are not restricted to *holonomic* constraints (2b) but may be applied as well to systems with *non-holonomic* constraints $\mathbf{0} = \mathbf{H}(\mathbf{q})\dot{\mathbf{q}}$ adding these new constraints to the hidden constraints (3a) at velocity level. Recent results show, however, some limitations in long-term simulations since there is no equivalent second order ODE in minimal coordinates for non-holonomic systems. General index reduction and solution techniques from DAE theory are used successfully for more complex applications like *servo constraints* in path following or *optimal control problems* for constrained systems [5].

Redundant constraints, Non-smooth constraints If constraint manifold $\mathfrak{M} := \{\,\mathbf{q} : \mathbf{g}(\mathbf{q}) = \mathbf{0}\,\}$ is non-empty but has a dimension less than the number m of holonomic constraints (2b) then these constraints are redundant, the constraint Jacobian $\mathbf{G}(\mathbf{q})$ is rank-deficient and the uniqueness of solutions $\big(\mathbf{q}(t), \boldsymbol{\lambda}(t)\big)$ is lost. However, the existence of a uniquely defined solution $\mathbf{q}(t)$ is still guaranteed substituting the equilibrium conditions (2a) by

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} - \mathbf{f}(t, \mathbf{q}, \dot{\mathbf{q}}) \in \text{range}\, \mathbf{G}^\top(\mathbf{q})\,,$$

see [6]. In more complex models with force vectors $\mathbf{f}(t, \mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\lambda})$ that contain friction terms depending on constraint forces and Lagrange multipliers, appropriate constraint forces are obtained considering joint clearances and force terms (1) with stiffness parameters ε representing the distribution of constraint forces in the real physical system. The combination of this approach with scalarization techniques from multi-objective optimization has also been used successfully to regularize a class of contact problems resulting in non differentiable constraint equations (2b).

References

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